QUALIFYING EXAMINATION, Part 1

1:00 PM – 4:00 PM, Thursday August 29, 2013

Attempt all parts of all four problems.

Please begin your answer to each problem on a separate sheet, write your 3 digit code and the problem number on each sheet, and then number and staple together the sheets for each problem. Each problem is worth 100 points; partial credit will be given.

Calculators may NOT be used.

Problem 1: Mathematical Methods

(a) (25 points) Form three orthonormal vectors $|I\rangle, |II\rangle, |III\rangle$ in terms of linearly independent vectors $|1\rangle, |2\rangle$ and $|3\rangle$ using the Gram-Schmidt procedure. What would go wrong if the vectors had been linearly dependent?

(b) (25 points) Consider the functions 1, x and x^2 in the interval $-1 \leq x \leq 1$. Form three orthonormal functions f_0, f_1 and f_2 that are linear combinations of these. The scalar product of f_i and f_j is defined to be $\int_{-1}^{1} f_i(x) f_j(x) dx$.

(c) (25 points) A particle in a cylinder of height H and radius R has energy

$$E = \frac{C^2}{R^2} + \frac{\pi^2}{H^2} \ ,$$

where C is a constant. Find the value of H/R that minimizes E for a fixed volume of the cylinder, first without and then with Lagrange multipliers.

(d) (25 points) Consider an N-dimensional normalized vector $|\psi\rangle$, and an $N \times N$ hermitian matrix A with eigenvectors $|\psi_n\rangle$ and eigenvalues λ_n:

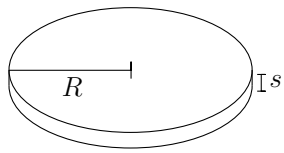
$$A|\psi_n\rangle = \lambda_n |\psi_n\rangle \ .$$

Assuming $\lambda_1 < \lambda_2 < \ldots < \lambda_N$, show that

$$\lambda_1 \leq \langle\psi|A|\psi\rangle \leq \lambda_N \ .$$

Problem 2: Classical Mechanics

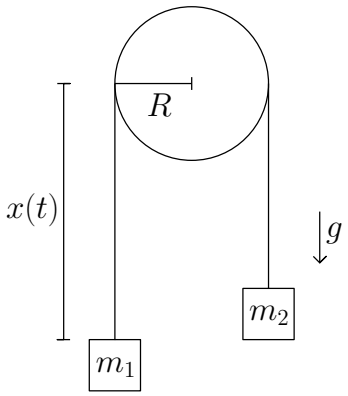
Consider a circular disc of radius R and infinitesimal thickness s (see figure), whose density varies with the radial distance as $\rho(r) = kr^2$.



(a) (15 points) Derive expressions for the total mass M of the disc and its moment of inertia I around its symmetry axis. Express I in terms of M and R.

(b) (10 points) Suppose the disc rotates at a constant angular velocity. What is the kinetic energy T of the disc in terms of M and the velocity v of the outer edge?

Now suppose that the disc is converted into a frictionless pulley. A string of length ℓ is run across it which connects two blocks with masses m_1 and m_2, where $m_1 > m_2$. Assume the string is massless, does not stretch, and does not slip on the pulley. The blocks can be treated as point-like, only move in the vertical direction, and are placed in a gravity field with a downward acceleration g (see figure).



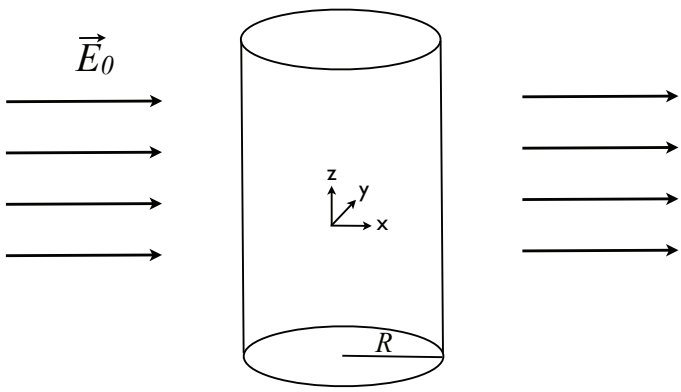
(c) (25 points) Write down a Lagrangian for this system in terms of the vertical distance $x(t)$ from the center of the pulley (measured positively downwards) to the block having mass m_1. You can assume that the potential energy vanishes at $x = 0$. Do you expect energy to be conserved in this system? Explain your answer.

(d) (25 points) Write down the Euler-Lagrange equation for $x = x(t)$ and derive an expression for the acceleration $\ddot{x}$. Express your result in terms of m_1, m_2, M, and g.

(e) (25 points) What is the tension T_1 in the string just above the block with mass m_1? What is the tension T_2 in the string just above the block with mass m_2? What is the net torque τ applied to the pulley? Express your results in terms of m_1, m_2, M, g and R.

Problem 3: Electromagnetism I

An infinitely long, grounded conducting hollow cylinder of radius R is placed into a uniform electric field $\vec{E}_0$. The field is perpendicular to the symmetry axis of the cylinder (see figure).



(a) (20 points) Using cylindrical coordinates r, ϕ, z, where the origin is chosen at the center of the cylinder and z is taken along the symmetry axis of the cylinder, argue that the electrostatic potential V is independent of z. What are the boundary conditions satisfied by the potential $V(r, \phi)$?

(b) (35 points) Find the electrostatic potential $V_{\text{out}}(r, \phi)$ outside the cylinder.

Hint: the general solution to Laplace's equation in polar coordinates is given by

$$V(r, \phi) = a_0 + b_0 \ln r + \sum_{m=1}^{\infty} (a_m r^m + b_m r^{-m})[A_m \sin(m\phi) + B_m \cos(m\phi)] \ ,$$

where a_m, b_m, A_m, B_m are constants.

(c) (20 points) What is the electrostatic potential inside the cylinder?

(d) (25 points) Find the surface charge density σ induced on the cylinder as a function of ϕ.

Problem 4: Electromagnetism II

An infinite dielectric slab of thickness d, assumed to be along the z axis, is subjected to a constant magnetic field $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$. A weak electromagnetic plane wave traveling in the z direction, $\mathbf{E}(z,t) = \mathbf{E}_0 e^{i(kz-\omega t)}$ (where $\mathbf{E}_0$ is a constant vector in the $x-y$ plane) impinges on this slab.

We model the slab by a collections of electrons (each of charge $-e$ and mass m) with uniform number density N that (in the absence of external fields) oscillate around their equilibrium positions with a spring constant $k = m\omega_0^2$. We assume the electrons to be non-interacting and ignore the magnetic component of the electromagnetic field. We also assume the z position of each electron to be frozen so that the electron's motion is restricted to the $x-y$ plane.

It is convenient to work out this problem using the unit vectors $(\mathbf{e}^+, \mathbf{e}^-, \hat{\mathbf{z}})$ where

$$\mathbf{e}^\pm = (\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})/\sqrt{2}\,.$$

Note that $\mathbf{e}^\pm \times \hat{\mathbf{z}} = \pm i\mathbf{e}^\pm$. Also, an arbitrary vector $\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}}$ in the x-y plane can be written as

$$\mathbf{A} = A^-\mathbf{e}^+ + A^+\mathbf{e}^-\,,$$

where $A^\pm \equiv (A_x \pm iA_y)/\sqrt{2}$.

(a) (20 points) Write down the classical equation of motion for a single such electron in terms of its displacement $\mathbf{r} = \mathbf{r}(t)$ in the x-y plane from its equilibrium position.

(b) (35 points) Find a solution of the equation of motion in (b) of the form

$$\mathbf{r}(z,t) = \mathbf{r}_0 e^{i(kz-\omega t)}\,,$$

where $\mathbf{r}_0$ is a constant vector in the x-y plane. Write your solution in terms of the cyclotron frequency $\omega_c = eB_0/mc$ (in Gaussian units; $\omega_c = eB_0/m$ in SI units).

Hint: decompose $\mathbf{r}_0$ and $\mathbf{E}_0$ in the basis of $\mathbf{e}^+$ and $\mathbf{e}^-$ and determine $r_0^\pm$ as a function of $E_0^\pm$.

(c) (25 points) Write an expression for the electric polarization $\mathbf{P}$ of the dielectric in terms of the displacement $\mathbf{r}$, the number density N and the electron's charge $-e$. Infer the effective dielectric constants $\epsilon^\pm$ that correspond to electromagnetic waves of $\pm$ polarizations, i.e., find $\epsilon^\pm$ such that the $\pm$ components of the displacement vector are given by $D^\pm = \epsilon^\pm E^\pm$.

(d) (20 points) Determine the indices of refraction $n^\pm$ for the two polarizations. Recall that index of refraction is c/v where v is the velocity in the medium, while c is the velocity in vacuum.

Explicit Forms of Vector Operations

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian $(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical (ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \mathbf{e}_3\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical (r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1 \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right]$$
$$+ \mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)\right]$$

35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

J	J	...
M	M	...
m_1 m_2		
m_1 m_2	Coefficients	
. .		
. .		
. .		

1/2×1/2

M	m_1 m_2	$J=1$	$J=0$
+1	+1/2 +1/2	1	
0	+1/2 −1/2	1/2	1/2
	−1/2 +1/2	1/2	−1/2
−1	−1/2 −1/2	1	

1×1/2

M	m_1 m_2	$J=3/2$	$J=1/2$
+3/2	+1 +1/2	1	
+1/2	+1 −1/2	1/3	2/3
	0 +1/2	2/3	−1/3
−1/2	0 −1/2	2/3	1/3
	−1 +1/2	1/3	−2/3
−3/2	−1 −1/2	1	

2×1/2

M	m_1 m_2	$J=5/2$	$J=3/2$
+5/2	+2 +1/2	1	
+3/2	+2 −1/2	1/5	4/5
	+1 +1/2	4/5	−1/5
+1/2	+1 −1/2	2/5	3/5
	0 +1/2	3/5	−2/5
−1/2	0 −1/2	3/5	2/5
	−1 +1/2	2/5	−3/5
−3/2	−1 −1/2	4/5	1/5
	−2 +1/2	1/5	−4/5
−5/2	−2 −1/2	1	

3/2×1/2

M	m_1 m_2	$J=2$	$J=1$
+2	+3/2 +1/2	1	
+1	+3/2 −1/2	1/4	3/4
	+1/2 +1/2	3/4	−1/4
0	+1/2 −1/2	1/2	1/2
	−1/2 +1/2	1/2	−1/2
−1	−1/2 −1/2	3/4	1/4
	−3/2 +1/2	1/4	−3/4
−2	−3/2 −1/2	1	

2×1

M	m_1 m_2	$J=3$	$J=2$	$J=1$
+3	+2 +1	1		
+2	+2 0	1/3	2/3	
	+1 +1	2/3	−1/3	
+1	+2 −1	1/15	1/3	3/5
	+1 0	8/15	1/6	−3/10
	0 +1	2/5	−1/2	1/10
0	+1 −1	1/5	1/2	3/10
	0 0	3/5	0	−2/5
	−1 +1	1/5	−1/2	3/10
−1	0 −1	2/5	1/2	1/10
	−1 0	8/15	−1/6	−3/10
	−2 +1	1/15	−1/3	3/5
−2	−1 −1	2/3	1/3	
	−2 0	1/3	−2/3	
−3	−2 −1	1		

3/2×1

M	m_1 m_2	$J=5/2$	$J=3/2$	$J=1/2$
+5/2	+3/2 +1	1		
+3/2	+3/2 0	2/5	3/5	
	+1/2 +1	3/5	−2/5	
+1/2	+3/2 −1	1/10	2/5	1/2
	+1/2 0	3/5	1/15	−1/3
	−1/2 +1	3/10	−8/15	1/6
−1/2	+1/2 −1	3/10	8/15	1/6
	−1/2 0	3/5	−1/15	−1/3
	−3/2 +1	1/10	−2/5	1/2
−3/2	−1/2 −1	3/5	2/5	
	−3/2 0	2/5	−3/5	
−5/2	−3/2 −1	1		

1×1

M	m_1 m_2	$J=2$	$J=1$	$J=0$
+2	+1 +1	1		
+1	+1 0	1/2	1/2	
	0 +1	1/2	−1/2	
0	+1 −1	1/6	1/2	1/3
	0 0	2/3	0	−1/3
	−1 +1	1/6	−1/2	1/3
−1	0 −1	1/2	1/2	
	−1 0	1/2	−1/2	
−2	−1 −1	1		

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

QUALIFYING EXAMINATION, Part 2

9:00 AM – 12:00 NOON, Friday August 30, 2013

Attempt all parts of all four problems.

Please begin your answer to each problem on a separate sheet, write your 3 digit code and the problem number on each sheet, and then number and staple together the sheets for each problem. Each problem is worth 100 points; partial credit will be given.

Calculators may NOT be used.

Problem 1: Quantum Mechanics I

For a spin-1/2 particle of mass m moving in a central potential $V(r)$ there is a "spin-orbit" interaction between the magnetic moment of the particle and the magnetic field that it sees as it moves spatially, leading to the Hamiltonian term

$$H_{LS} = \frac{1}{2mc^2}\,\frac{1}{r}\,\frac{dV(r)}{dr}\,\vec{L}\cdot\vec{S}\ ,$$

where $\vec{L}$ and $\vec{S}$ are, respectively, the orbital angular momentum and spin of the particle.

In the absence of this term, the energy eigenfunctions of the system may be written in the form

$$R_{nl}(r)\ Y_l^{m_l}(\theta,\phi)\ \chi_\pm\ ,$$

where $\chi_\pm$ are two-component spinors describing a spin up/spin down particle, and n is a principal quantum number. These energy eigenfunctions are also eigenfunctions of $\vec{L}^2$, L_z, $\vec{S}^2$, and S_z.

(a) (20 points) In the absence of H_{LS}, and for a general potential $V(r)$, the energy depends on n and l. (For the special case of a Coulomb potential, the energy depend on only n.) In the general case, what is the degeneracy of each level? Why is this degeneracy present? What are the possible values of the total angular momentum j of this system?

(b) (30 points) The energy shift due to H_{LS} may be expressed to first order in (degenerate) perturbation theory as the expectation value of H_{LS} with respect to normalized linear combinations of the above, degenerate eigenfunctions for which $\vec{L}^2$, $\vec{S}^2$, $\vec{J}^2$ and J_z are diagonal. (You do not need to write these linear combinations explicitly.) Write an expression for the expectation value of $\vec{L}\cdot\vec{S}$ in each of the possible total angular momentum states.

(c) (30 points) Using the result in (b), derive an expression for the energy shift of the energy eigenstates in terms of their orbital angular momentum l and the radial integral

$$\left\langle \frac{1}{r}\,\frac{dV(r)}{dr}\right\rangle_{nl} \equiv \int_0^\infty R_{nl}(r)\frac{1}{r}\,\frac{dV(r)}{dr}r^2dr\ .$$

How many new energy levels appear? What is the degeneracy of each level? Why does this degeneracy remain?

(d) (20 points) Apply this result to the sodium atom for which, in the ground state, the inner 10 electrons fill up all the $n=1$ and $n=2$ levels, and the 11th electron is in the 3s ($l=0$) state. Focus on just this electron, moving in an effective potential $V(r)$ created by the nucleus and the inner 10 electrons. Are the 3s and 3p ($l=1$) levels degenerate in the absence of H_{LS}? In the presence of H_{LS}, into how many levels is the $3p$ level split? What are the possible j values? Assuming that the above radial integral is positive, how are the corresponding energy levels ordered with respect to j?

Problem 2: Quantum Mechanics II

Consider a one-dimensional harmonic oscillator with mass m and frequency ω that is described by the annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$, respectively. These operators satisfy the commutation relation $[a, a^\dagger] = 1$.

(a) (20 points) If α is a complex number, show that the unitary operator $\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$ translates the annihilation operator $\hat{a}$ by α, i.e.

$$\hat{D}^\dagger(\alpha)\,\hat{a}\,\hat{D}(\alpha) = \hat{a} + \alpha\ .$$

Hint:

$$e^{\hat{A}}\hat{X}e^{-\hat{A}} = \hat{X} + [\hat{A}, \hat{X}] + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{X}]] + \ldots$$

for any two operators $\hat{A}$ and $\hat{X}$.

(b) (25 points) Consider the state (called a "coherent state") defined as $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$, where $|0\rangle$ is the oscillator's ground state). Expand this state in the basis of energy eigenstates $|n\rangle$ $(n = 0, 1, 2, \ldots)$.

Hint: for two operators $\hat{A}$ and $\hat{B}$ whose commutator is a c-number, we have

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]}\ .$$

Use this formula for $\hat{A} = \alpha\hat{a}^\dagger$ and $\hat{B} = -\alpha^*\hat{a}$.

(c) (25 points) Show that the coherent state $|\alpha\rangle$ defined in (b) is an eigenstate of the operator $\hat{a}$. Find its eigenvalue.

Hint: you can solve this part by using either (a) or (b).

(d) (30 points) Calculate the time dependence of a state initially prepared in the coherent state $|\alpha\rangle$. Show that this state is (up to phase) a coherent state $|\alpha(t)\rangle$ and write an explicit expression for $\alpha(t)$.

Problem 3: Statistical Mechanics I

Consider a spinless particle of mass m. The particle can move freely in the x-y plane within a square of area A, but along the z direction it is subjected to a harmonic potential $\frac{1}{2}m\omega^2 z^2$. The particle is at equilibrium at temperature T.

(a) (35 points) At sufficiently high temperatures T, this particle can be described classically. Calculate the classical partition function Z of the particle. Then use Z to calculate its average energy E, free energy F and entropy S as a function of temperature.

Compare your result for the average energy E with what you find from the equipartition theorem.

Useful integral:

$$\int_{-\infty}^{\infty} du \exp(-au^2) = \sqrt{\frac{\pi}{a}} \qquad \text{for } a > 0 \ .$$

(b) (20 points) Estimate the temperature T_1 below which quantum mechanical effects become relevant in the z direction. Then estimate the temperature T_2 below which quantum mechanical effects become relevant in the x-y plane.

In the following, assume A is sufficiently large such that $T_2 \ll T_1$.

(c) (30 points) Find the average energy E of the particle for temperatures T such that $T_2 \ll T < T_1$.

(d) (15 points) Find the energy E and the entropy S of the particle at $T = 0$.

Problem 4: Statistical Mechanics II

Consider an ideal gas of neutrinos in a box of volume V. Neutrinos are spin 1/2 particles and for this problem assume they are massless.

(a) (15 points) Find the density of single-particle states $g(\epsilon)$ of the neutrinos at energy ϵ. How does this density of states compare to the density of states of a photon gas? Explain your answer.

(b) (30 points) Assuming the total number of neutrinos is not conserved (as is the case of the photon gas), calculate the energy density E/V of the neutrino gas at temperature T. You can express your result in terms of a dimensionless definite integral that is temperature-independent.

(c) (30 points) Find the ratio between the energy density of the neutrino gas and that of the photon gas at the same temperature T.

Useful integrals:

$$\begin{aligned}\int_0^\infty dx \frac{x^{n-1}}{e^x+1} &= (1-2^{1-n})\Gamma(n)\zeta(n) \\ \int_0^\infty dx \frac{x^{n-1}}{e^x-1} &= \Gamma(n)\zeta(n) \ ,\end{aligned}$$

where n is an integer and $\zeta(x)$ is the Riemann zeta function.

(d) (25 points) Determine the equation of state of the neutrino gas expressing the relation between its pressure and energy density.

Hint: a possible derivation is based on the following expression for the pressure p:

$$p = \sum_r \left(-\frac{\partial \epsilon_r}{\partial V}\right) \langle n_r \rangle \ ,$$

where ϵ_r and $\langle n_r \rangle$ are, respectively, the energy and average occupation number of the single-particle state r. The dependence of ϵ on V is determined through the dispersion relation (expressing the single-particle energy in terms of the momentum $\hbar k$) and the quantization condition on $\mathbf{k}$.

Explicit Forms of Vector Operations

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian $(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical (ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \mathbf{e}_3\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical (r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1 \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right]$$
$$+ \mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)\right]$$

35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

J	J	...
M	M	...
m_1 m_2		
m_1 m_2	Coefficients	
. .		
. .		
. .		

1/2×1/2

M	m_1 m_2	$J=1$	$J=0$
+1	+1/2 +1/2	1	
0	+1/2 −1/2	1/2	1/2
	−1/2 +1/2	1/2	−1/2
−1	−1/2 −1/2	1	

1×1/2

M	m_1 m_2	$J=3/2$	$J=1/2$
+3/2	+1 +1/2	1	
+1/2	+1 −1/2	1/3	2/3
	0 +1/2	2/3	−1/3
−1/2	0 −1/2	2/3	1/3
	−1 +1/2	1/3	−2/3
−3/2	−1 −1/2	1	

2×1/2

M	m_1 m_2	$J=5/2$	$J=3/2$
+5/2	+2 +1/2	1	
+3/2	+2 −1/2	1/5	4/5
	+1 +1/2	4/5	−1/5
+1/2	+1 −1/2	2/5	3/5
	0 +1/2	3/5	−2/5
−1/2	0 −1/2	3/5	2/5
	−1 +1/2	2/5	−3/5
−3/2	−1 −1/2	4/5	1/5
	−2 +1/2	1/5	−4/5
−5/2	−2 −1/2	1	

3/2×1/2

M	m_1 m_2	$J=2$	$J=1$
+2	+3/2 +1/2	1	
+1	+3/2 −1/2	1/4	3/4
	+1/2 +1/2	3/4	−1/4
0	+1/2 −1/2	1/2	1/2
	−1/2 +1/2	1/2	−1/2
−1	−1/2 −1/2	3/4	1/4
	−3/2 +1/2	1/4	−3/4
−2	−3/2 −1/2	1	

2×1

M	m_1 m_2	$J=3$	$J=2$	$J=1$
+3	+2 +1	1		
+2	+2 0	1/3	2/3	
	+1 +1	2/3	−1/3	
+1	+2 −1	1/15	1/3	3/5
	+1 0	8/15	1/6	−3/10
	0 +1	2/5	−1/2	1/10
0	+1 −1	1/5	1/2	3/10
	0 0	3/5	0	−2/5
	−1 +1	1/5	−1/2	3/10
−1	0 −1	2/5	1/2	1/10
	−1 0	8/15	−1/6	−3/10
	−2 +1	1/15	−1/3	3/5
−2	−1 −1	2/3	1/3	
	−2 0	1/3	−2/3	
−3	−2 −1	1		

3/2×1

M	m_1 m_2	$J=5/2$	$J=3/2$	$J=1/2$
+5/2	+3/2 +1	1		
+3/2	+3/2 0	2/5	3/5	
	+1/2 +1	3/5	−2/5	
+1/2	+3/2 −1	1/10	2/5	1/2
	+1/2 0	3/5	1/15	−1/3
	−1/2 +1	3/10	−8/15	1/6
−1/2	+1/2 −1	3/10	8/15	1/6
	−1/2 0	3/5	−1/15	−1/3
	−3/2 +1	1/10	−2/5	1/2
−3/2	−1/2 −1	3/5	2/5	
	−3/2 0	2/5	−3/5	
−5/2	−3/2 −1	1		

1×1

M	m_1 m_2	$J=2$	$J=1$	$J=0$
+2	+1 +1	1		
+1	+1 0	1/2	1/2	
	0 +1	1/2	−1/2	
0	+1 −1	1/6	1/2	1/3
	0 0	2/3	0	−1/3
	−1 +1	1/6	−1/2	1/3
−1	0 −1	1/2	1/2	
	−1 0	1/2	−1/2	
−2	−1 −1	1		

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$