Colloquium, Yale University
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Generalized Global Symmetries

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Symmetry

- Symmetry has proven, from time and again, to be of fundamental importance for describing Nature.
- In recent years, there has been a revolution in our understanding of symmetry. In particular, the notion of global symmetry has been generalized in different directions.
- These generalized global symmetries are some of the few **universally applicable** tools to analyze general quantum systems.

Generalized global symmetries

- These new symmetries lead to several surprising consequences:
  - new constraints on renormalization group flows
  - new implications for the phase diagram of gauge theories
  - new organizing principles of quantum phases of matter
  - new interpretations for real-world physics (e.g., pion decay)
  - …
- Active collaboration between experts from high energy physics, condensed matter physics, quantum gravity, and mathematics.

What is symmetry?

- *But what is symmetry?*

- What is a global symmetry and what is a gauge symmetry?

Gauge "symmetry"

- Gauge (local) symmetry is powerful. It makes manifest the Lorentz invariance, unitarity, locality of a theory.
- It is the language of particle physics: $SU(3)\times SU(2)\times U(1)_Y$ gauge symmetry.
- Diffeomorphism in General Relativity.
- However, the Hilbert space is gauge-invariant because of the Gauss law.
- Gauge "symmetry" does not act on anything. It's a redundancy.
- In noncompact space or space with boundary, the notion of gauge symmetry can be made more invariant by fixing a boundary condition.

Gauge redundancy

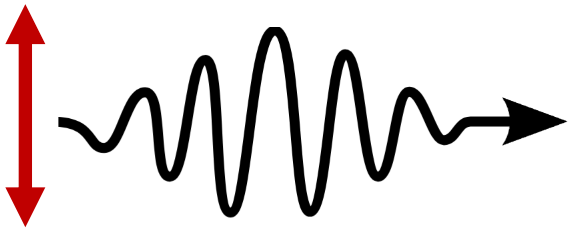
- Moreover, given a single quantum system, there can be a gauge symmetry in one description but not in another.
- For example, a free photon in 2+1d has one transverse polarization, and is therefore dual to a free scalar:

Gauge symmetry

$$B \leftrightarrow \partial_0 \phi$$
$$E_x \leftrightarrow \partial_y \phi$$
$$E_y \leftrightarrow -\partial_x \phi$$

No gauge symmetry

- This is common in condensed matter physics. For example, the particle-vortex duality.



Global symmetry

- In contrast, global symmetry acts nontrivially on the Hilbert space.
- It is an intrinsic property of a quantum system that should matched under duality.
- Global symmetries can have anomalies, i.e., obstruction to gauging them. They lead to 't Hooft anomaly matching conditions.
- It's important to characterize global symmetries abstractly and invariantly, without referring to any Lagrangian description.
- So, what is a global symmetry?

What is global symmetry?

1. It’s a transformation of the fields that leaves the Lagrangian invariant, $\mathcal{L}(\phi + \delta\phi) = \mathcal{L}(\phi)$.

   But what if the quantum system doesn’t have a Lagrangian?

2. It’s an operator that commutes with the Hamiltonian, $[U, H] = 0$.

   Einstein would beg to differ: why is the time direction distinguished?

3. It’s defined by a conservation equation, e.g., $\partial_0 \rho = \partial_i j^i$.

   But what about discrete symmetries (e.g., CP)?

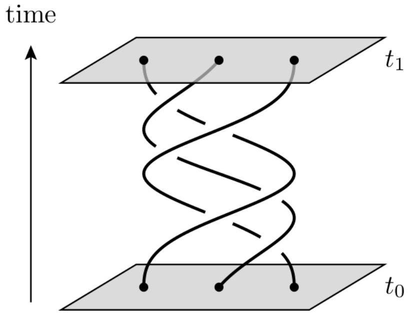
4. Do symmetries have to be (anti-)unitary?

   Wigner said so. But really?

- No definitive and universal answers. An ongoing conversation.

Generalized global symmetries

Many other generalizations of global symmetries not discussed here, e.g., dipole symmetry, asymptotic symmetry,…

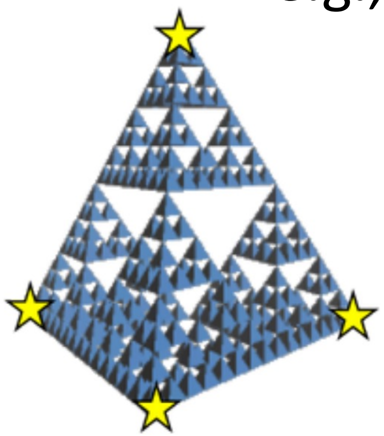


Higher-form symmetries

e.g., center symmetry, abelian anyons

Subsystem symmetries

e.g., fractons



Non-invertible symmetries

e.g., Ising model, non-abelian anyons, QED, QCD,…

$$\pi^0 \rightarrow \gamma\gamma$$

Noether current

- Consider a QFT with a conserved Noether current

$$\partial^\mu j_\mu = 0.$$

- The $U(1)$ symmetry operator is

$$U_\vartheta = \exp(i\vartheta \int d^3x\, j_0)$$

- Thanks to the conservation equation, it is conserved

$$\partial_0 U_\vartheta = 0$$

- Quantum mechanically, $[H, U_\vartheta] = 0$.

Symmetry and topology

- For relativistic systems in Euclidean signature, the time direction is on the same footing as any other spatial direction [Einstein 1905].
- We can therefore integrate the current on a general closed (codim-1) 3-manifold $M^{(3)}$ in 4-dimensional Euclidean spacetime:

$$\exp(i\vartheta \int d^3x\, j_0)$$

$$\downarrow$$

$$U_\vartheta\left(M^{(3)}\right) = \exp(i\vartheta \oint_{M^{(3)}} dn^\mu j_\mu)$$

- The conservation equation $\partial_0 U_\vartheta = 0$ is now **upgraded** to the fact that $U_\vartheta(M^{(3)})$ depends on $M^{(3)}$ only topologically (divergence theorem).

Generalized global symmetries

Properties of symmetry op.	Ordinary symmetry	Higher-form symmetry	Subsystem symmetry	Non-invertible symmetry
Codimension in spacetime	1	> 1	> 1	≥ 1
Topological	yes	yes	not completely but conserved in time	yes
Fusion rule	group $g_1 \times g_2 = g_3$	group $g_1 \times g_2 = g_3$	group $g_1 \times g_2 = g_3$	category $\mathcal{D} \times \mathcal{D}^\dagger \neq 1$

Next, we generalize the ordinary global symmetry by modifying these conditions.

Generalized global symmetries

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More generalizations by combining different columns!

Higher-Form Symmetry

Global symmetries and generalizations

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1-form symmetries

[Gaiotto-Kapustin-Seiberg-Willett 2014,…]

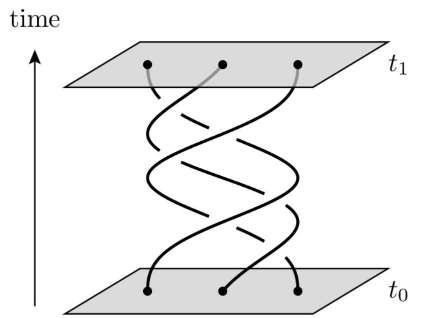
- Ordinary (0-form) global symmetries acts on point operators:
$$\Phi(x) \to \exp(i\alpha)\,\Phi(x)$$
- 1-form global symmetries can act on the gauge field
$$A_\mu(x) \to A_\mu(x) + \beta_\mu(x) \quad , \quad \partial_\mu \beta_\nu(x) - \partial_\nu \beta_\mu(x) = 0$$
- More invariantly, it acts on the gauge-invariant Wilson line:
$$\exp(i\oint A) \to \exp(i\oint \beta)\,\exp(i\oint A)$$
- The 1-form symmetry charge is the electric flux surface (codim-2 in 3+1d)
$$\oint_{M^{(2)}} \vec{E}\cdot d\vec{n}$$
- In pure $SU(N)$ gauge theory, the $\mathbb{Z}_N$ center symmetry is a 1-form symmetry.

Higher-form symmetries and anomalies

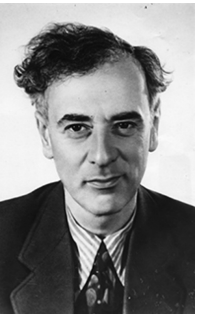
[Gaiotto-Kapustin-Seiberg-Willett 2014,…]

- Higher-form global symmetries can also have anomalies.
- Nontrivial anomalies imply that the low energy phase can **NOT** be trivially gapped with a non-degenerate ground state and no topological dof.
- Example: 3+1d $SU(2)$ pure gauge theory at $\theta = \pi$ has a mixed anomaly between CP and the $\mathbb{Z}_2$ one-form center symmetry. The low energy phase can **NOT** be a trivially confining phase. [Gaiotto-Kapustin-Komargodski-Seiberg 2017]
- In contrast, we expect the $\theta = 0$ phase to be a trivially confining phase [1 million dollar from Clay Mathematics Institute].

Generalized Landau paradigm



- Landau paradigm: phases of matter are classified by how they represent the symmetries.
- Apparent exceptions include of topological order [Wen, …] that seemingly has no symmetry.
- Abelian anyons in the topological order generate 1-form global symmetries in the low energy Chern-Simons theory.
- Topological order as a spontaneous symmetry breaking phase of a 1-form global symmetry – Landau is right after all!



Non-abelian anyons [Moore-Read 1991,…] generate non-invertible 1-form symmetries.

Subsystem Symmetry

Generalized global symmetries

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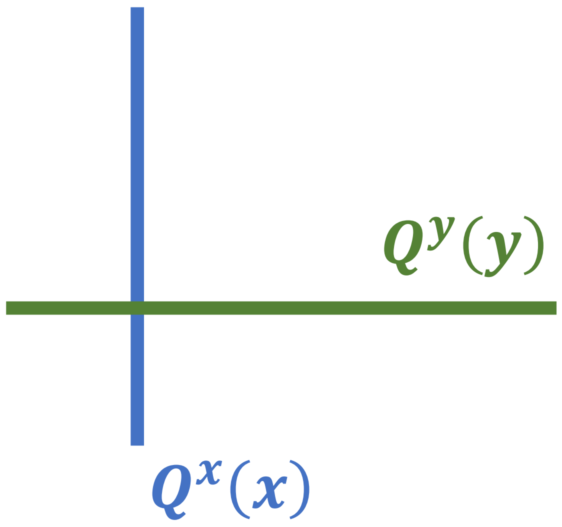
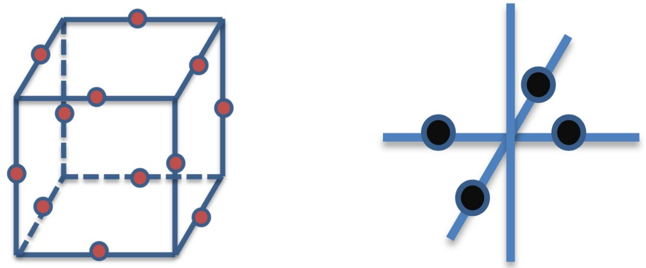
Subsystem symmetry

- There are many interesting lattice models, such as fractons, exhibiting subsystem symmetries.
- The subsystem symmetry charges are supported on certain higher-codimensional loci in space (e.g., straight lines on a plane) [..., Paramekanti-Balent-Fisher 2002, ...]. They depend **NOT** only on the topology of the manifolds.
- The number of subsystem symmetry charges generally depends on the number of lattice points.
- Low energy observables are sensitive to short distance details: UV/IR mixing [Seiberg-Shao 2020, Gorantla-Lam-Seiberg-SHS 2021].

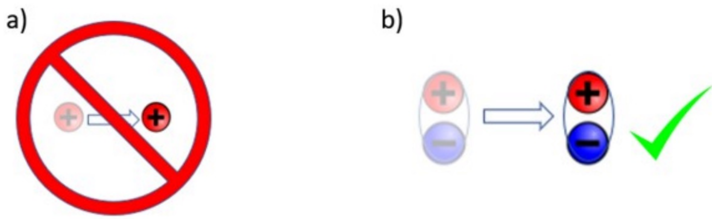


Fractons



- Fractons [Chamon 2005, Haah 2011,…] are a large class of gapped lattice spin models with many peculiar features.
- They do **not** admit a conventional continuum field theory limit. Challenge the canonical paradigm that QFT describes low energy phases.
- Example: the 3+1d X-cube model [Vijay-Haah-Fu 2016]:

1. Robust ground state degeneracy that grows subextensively: GSD = 2^{6L-3} where L is the number of lattice sites in every direction. It becomes infinite in the continuum limit, reflecting UV/IR mixing.
2. Excitations have restricted mobility.



Space-like and time-like symmetries

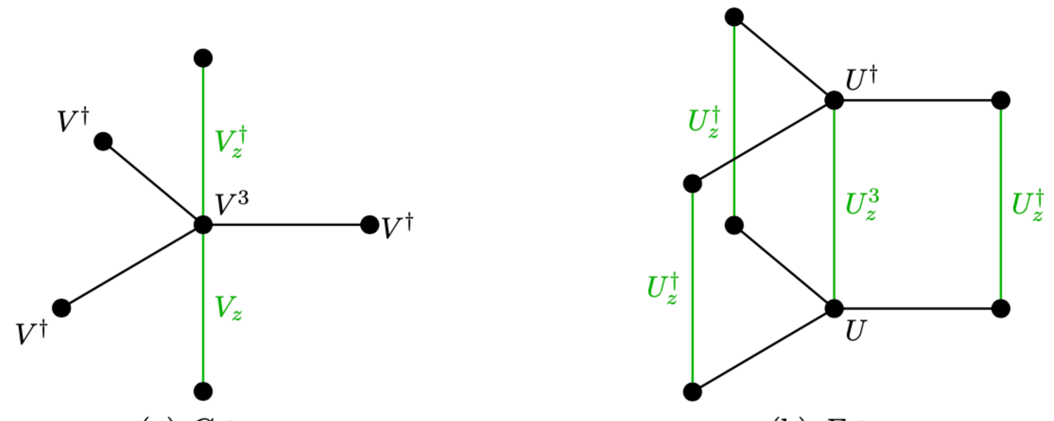
[Gorantla-Lam-Seiberg-SHS 2022]

Fracton Peculiarities	Symmetry Explanations
Ground State Degeneracy	Space-like subsystem symmetries and their anomalies Act on states
Restricted Mobility	Time-like subsystem symmetries Act on defects

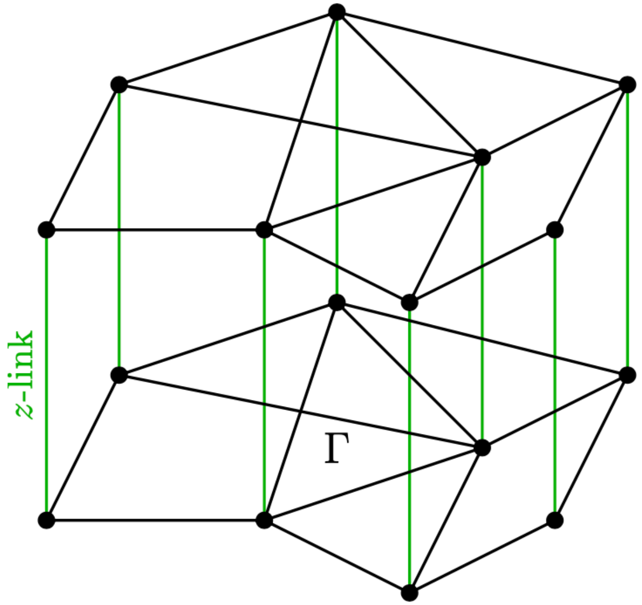
Fractons on graphs

[Gorantla-Lam-SHS-Seiberg 2022]

- There are even weirder global symmetries in other exotic models motivated by condensed matter systems [Haah 2011, Yoshida 2013, Ma et al. 2020,…].
- New gapped $\mathbb{Z}_N$ fracton/lineon lattice models on a general spatial graph Γ (times a line) [Gorantla-Lam-SHS 2022, Ebisu-Han 2022, Gorantla-Lam-Seiberg-SHS 2022].
- Analogous to defining QFT on general curved spacetime.



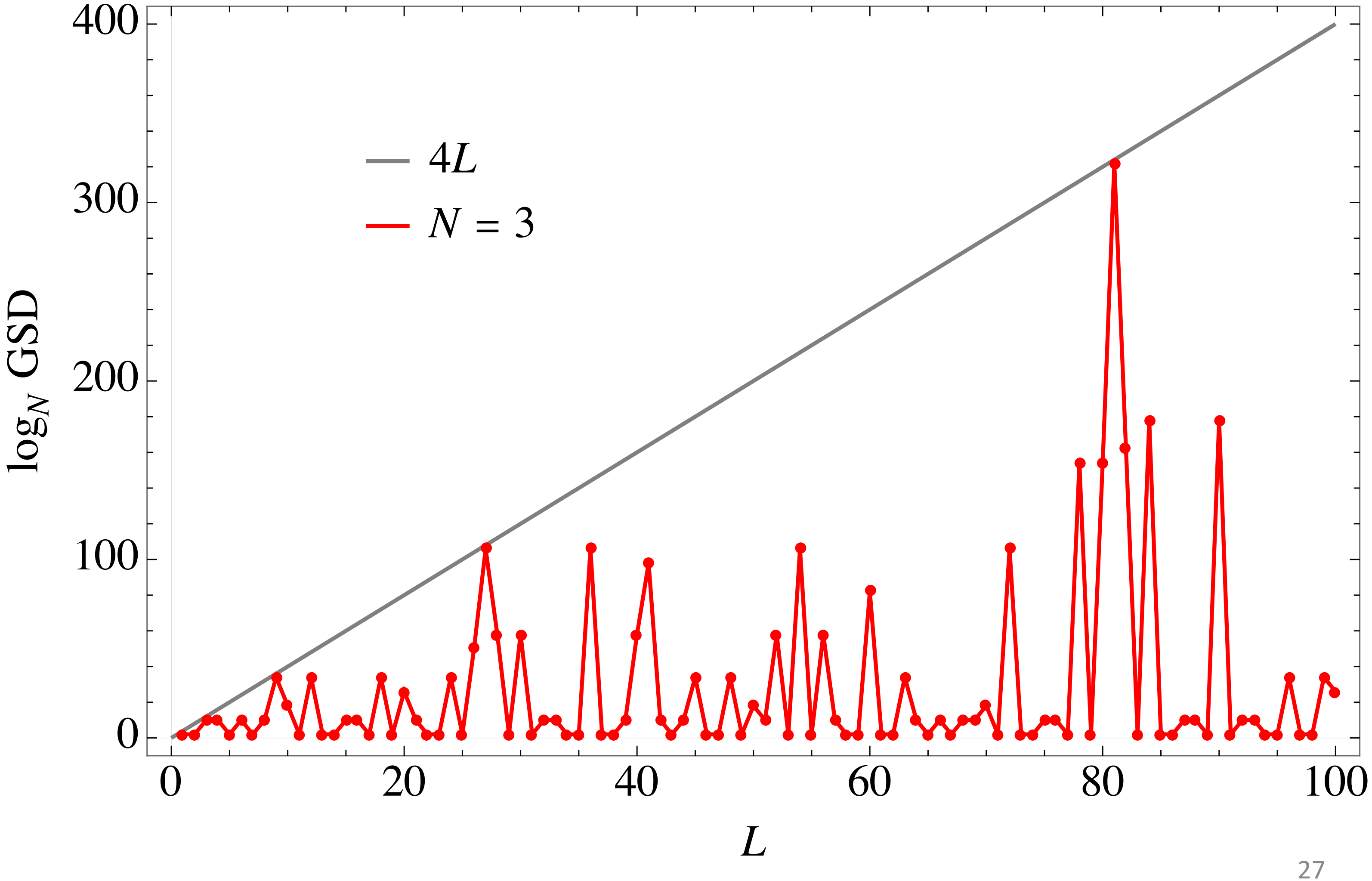
(a) G-term (b) F-term



Fractons on graphs

[Gorantla-Lam-Seiberg-SHS 2022]

- Take the spatial lattice to be a $L \times L \times L$ cubic lattice.
- The ground state degeneracy (GSD) depends on L in an erratic way.



What's done cannot be undone:

Non-invertible Symmetry

Generalized global symmetries

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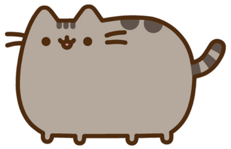
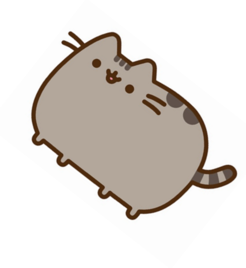
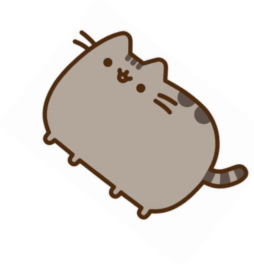
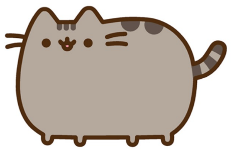
A first look into non-invertible symmetry

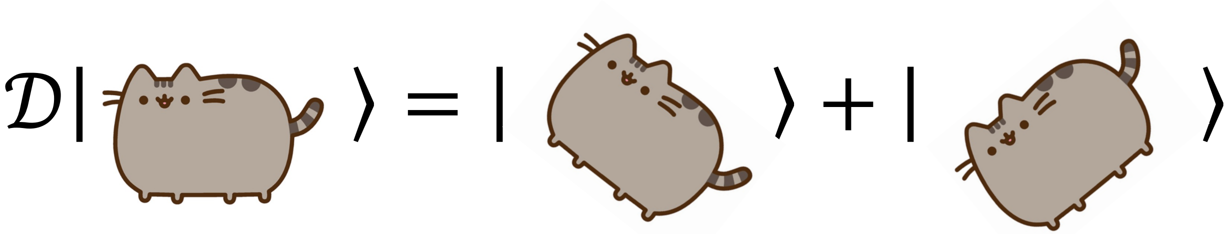
- Ordinary global symmetry is invertible. For example, consider a rotation operator U_θ by an angle θ:

$U_\theta|$  $\rangle = |$  $\rangle$

$U_{-\theta}|$  $\rangle = |$  $\rangle$

"What's done cannot be undone."

- In quantum systems, we can have superposition of quantum states. Schroedinger's cat can be both alive and dead.
- Let $\mathcal{D} = U_\theta + U_{-\theta}$



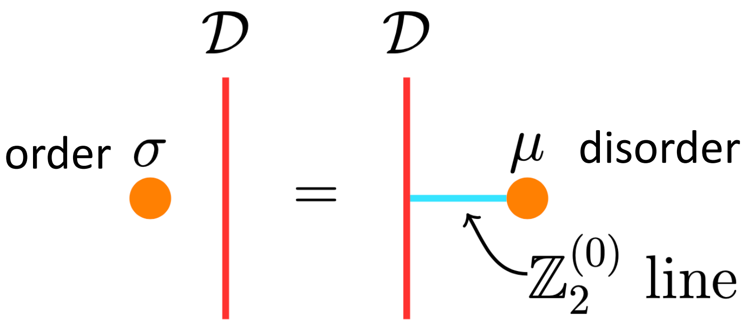
- $\mathcal{D}$ is not invertible; you get more and more cats every time you act with $\mathcal{D}$.
- In some systems, the unitary U_θ itself is **NOT** well-defined, but $\mathcal{D} = U_\theta + U_{-\theta}$ is (e.g., from a discrete gauge symmetry).
- In this case, $\mathcal{D}$ is a topological (in particular, conserved) operator that is genuinely non-invertible. It's not made out of unitaries.
- Not all non-invertible symmetries are of this kind.

Non-invertible symmetries

Why should we think of the non-invertible operators as generalized global symmetries?

- It leads to new conservation laws and selection rules.
- Some non-invertible lines can be gauged [Brunner-Carqueville-Plencner 2014].
- They can have generalized anomalies, which lead to generalized 't Hooft anomaly matching conditions. They result in nontrivial constraints on the renormalization group flows [Chang-Lin-SHS-Wang-Yin 2018].
- This inclusion consolidates [Rudelius-SHS 2020, Heidenreich et al. 2021] conjectures about the absence of global symmetry in quantum gravity [Misner-Wheeler 1957, Polchinski 2004, Banks-Seiberg 2010].

Non-invertible symmetry



- In the recent years, there has been rapid developments of non-invertible global symmetry [Bhardwaj-Tachikawa 2017, Tachikawa 2017, Chang-Lin-SHS-Yin-Wang 2018,…, Choi-Cordova-Hsin-Lam-SHS 2021, Kaidi-Ohmori-Zheng 2021,…].
- It is **NOT** implemented by a unitary operator. It also cannot be expressed in terms of sums of unitaries.

$$\mathcal{D}\times\mathcal{D}^\dagger \neq 1$$

- It is discovered in a variety of quantum systems, including Ising model (Kramers-Wannier duality line), CFT, spin chains, axions, free Maxwell theory, Yang-Mills theory, …
- New global symmetry in the real-world QED and QCD! [My seminar tomorrow]
- Mathematical conceptualization [Freed-Moore-Teleman 2022].

Non-invertible symmetry in Nature

[Choi-Lam-SHS 2022, Cordova-Ohmori 2022][My seminar tmr]

- In 3+1d massless QED, the classical chiral symmetry $U(1)_A$

$$\Psi \rightarrow \exp(i\theta\gamma_5)\Psi$$

is not completely broken by the Adler-Bell-Jackiw [1969] anomaly. Rather, it is resurrected as an infinite non-invertible global symmetry labeled by the rational numbers.

- The chiral symmetry is "**cured**" by the 2+1d Fractional Quantum Hall States.

$$\mathcal{D}_{1/N}(M) \equiv \exp[\oint_M (\frac{2\pi i}{2N} \star j^A + \frac{iN}{4\pi} ada + \frac{i}{2\pi} adA)]$$

- In QCD, it gives an alternative explanation for the neutral pion decay

$$\pi^0 \rightarrow \gamma\gamma$$

- *Pion decays because of the non-invertible symmetry!*

History of chiral symmetry in QED

Year

It **IS** a (non-invertible) symmetry!

2022

It is **NOT** a symmetry

ABJ 1969

It is a symmetry

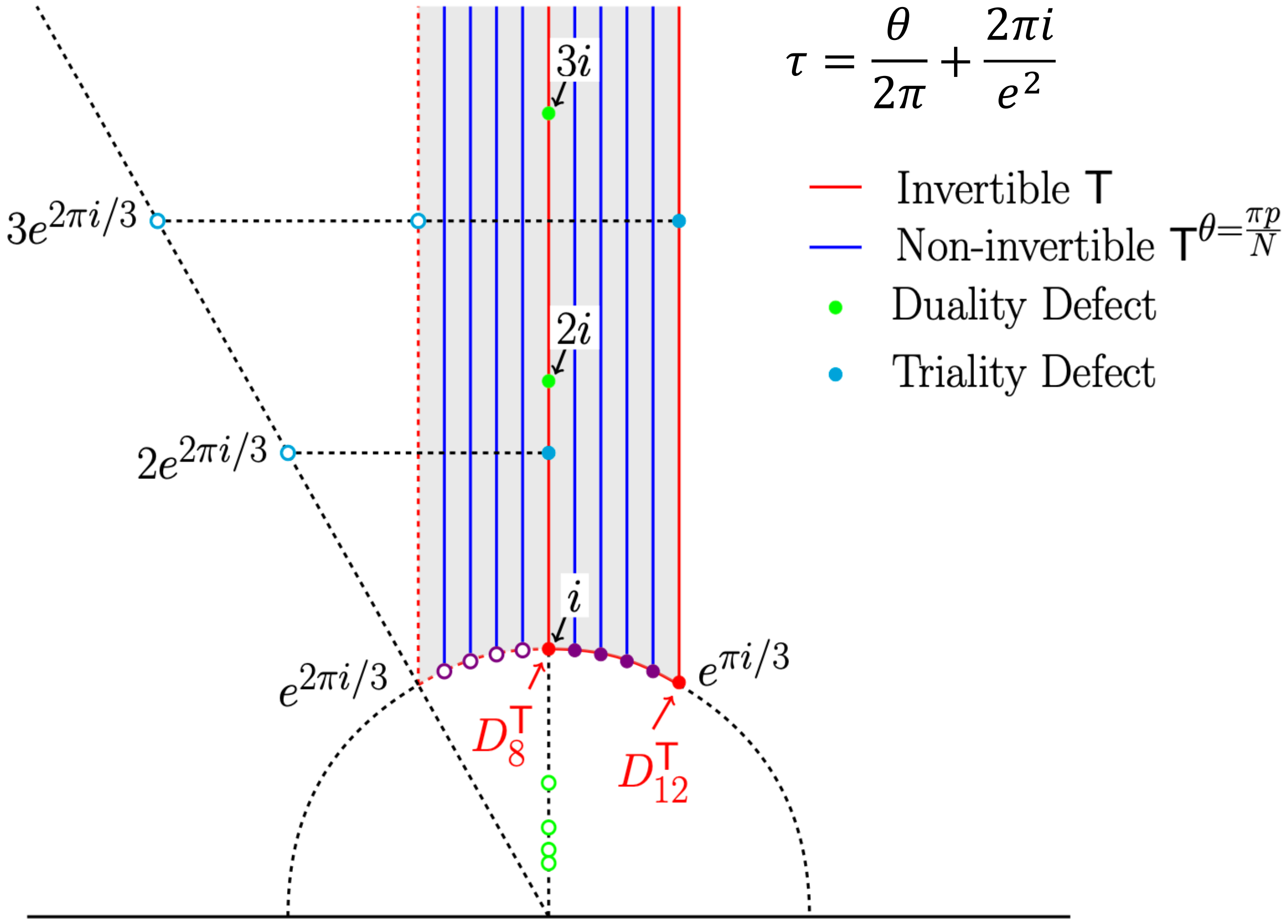
Non-invertible CP symmetry

[Choi-Lam-SHS 2022]

- It is commonly stated that CP or T is violated whenever the θ-angle is neither 0 or π.
- $U(1)$ gauge theory is time-reversal invariant for every rational θ angle

$$\theta = \frac{\pi p}{N}$$

- Non-invertible CP and time-reversal symmetry



Conclusion

- We have discussed three generalizations of global symmetries, higher form symmetries, subsystem symmetries, and non-invertible symmetries. Many other generalizations.
- This more general perspective of global symmetry unifies many known phenomena into a coherent framework.
  - Generalized global symmetries and their anomalies provide an invariant characterization of many topological phases of matter such as fractons.
  - Generalized Landau paradigm.
- More importantly, they lead to new dynamical consequences that are otherwise obscured.
  - Generalizations of the ‘t Hooft anomaly matching condition lead to nontrivial constraints on renormalization group flows.
- **New** symmetries in **new** and **old** QFTs, including our Nature!

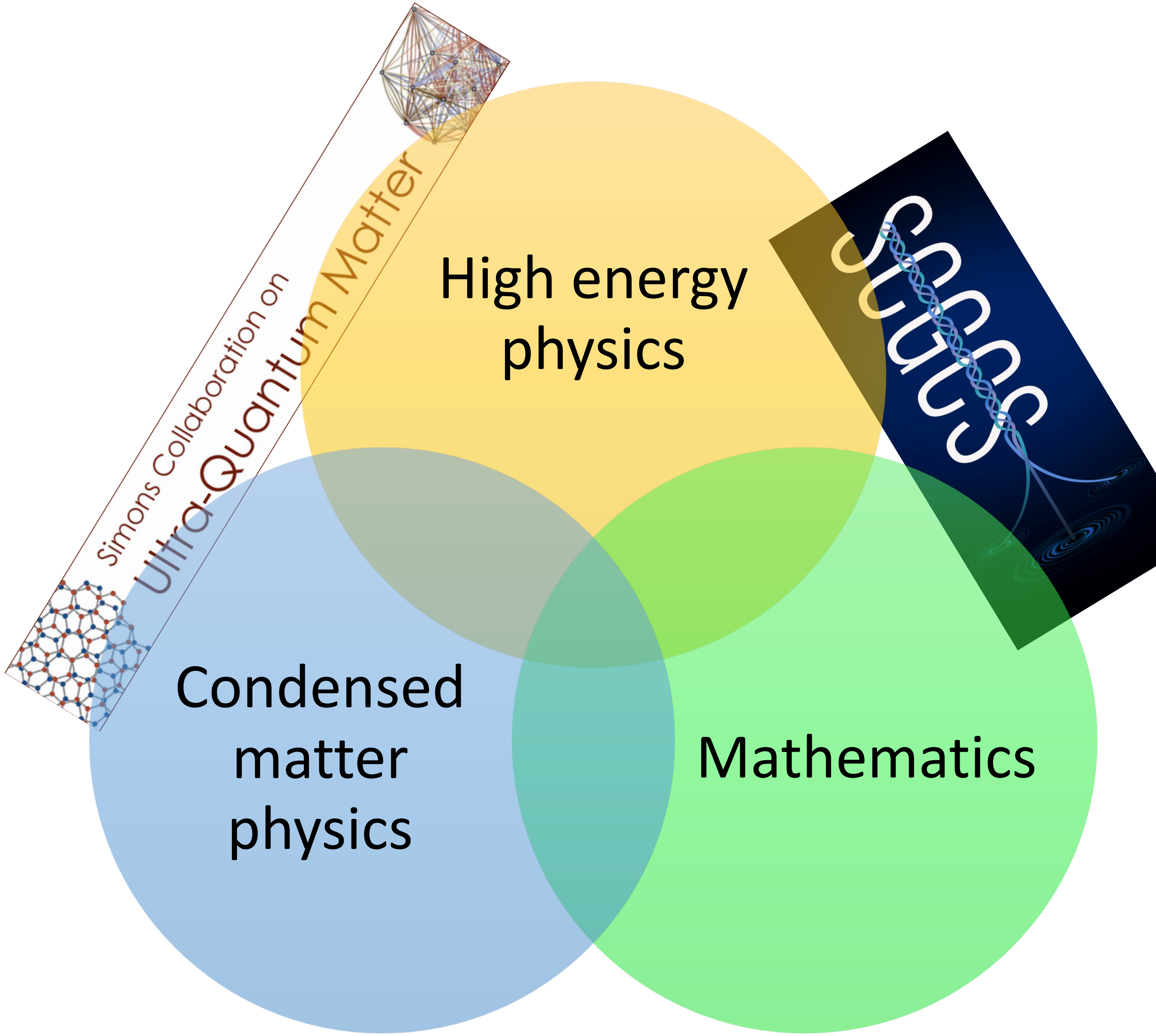
Outlook

- What qualifies as a symmetry?
- Are there more new global symmetries in the Standard Model? Are they useful?
- New symmetries for the hierarchy problems and naturalness problems.
- New time-reversal symmetries even at nonzero θ-angle. New insights into the strong CP problem?
- How do we use these symmetries to organize and even classify quantum phases of matter?

Activities

- TASI 2023 (co-organizer):
  Aspect of Symmetry

- Aspen Workshop 2023 (co-organizer):
  Traversing the Particle Physics Peaks - Phenomenology to Formal



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Applications	Hydrogen,…	Anyons,…	Fractons,…	Pions, axions, non-abelian anyons…

Thank you for listening!

Non-invertible symmetry

1960s	1988-	2010	2017-	2021	2022 Mar	2022 May	
Conserved charges in integrable systems	Non-inv lines in 1+1d RCFT 2+1d TQFT	Non-inv surfaces in 2+1d TQFT	Non-inv op as generalized sym Constraints on RG in 1+1d	3+1d gauge theories	Non-inv sym from higher-form sym	Non-inv sym in Nature	???

Above I mostly focus on codim-1 non-inv op.
Many many other developments not listed.