**QUALIFYING EXAMINATION, Part 1**

**9:00 – 11:30 am, Thursday September 2, 2021**

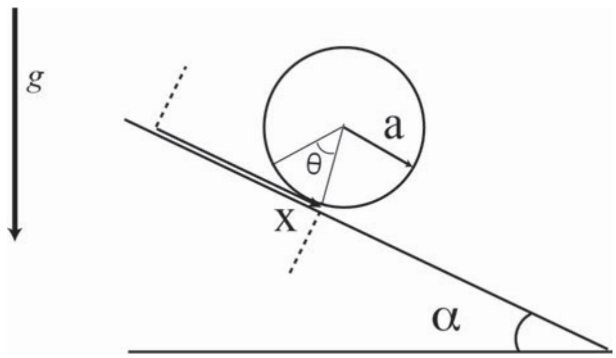
**Attempt all parts of both problems.**

Please begin your answer to each problem on a separate sheet, write your 3 digit code and the problem number on each sheet, and then number and staple together the sheets for each problem. Each problem is worth 100 points; partial credit will be given.

Calculators and cell phones may NOT be used.

**Problem 1: Classical Mechanics I**

Consider a uniform disk of of mass $M$ and radius $a$ rolling without sliding down an incline plane (see figure). $x$ is the distance traveled by the disk along the incline, $\theta$ is the rotation angle of the disk, and $\alpha$ is the tilt angle of the incline.




(a) (20 points) Write the Lagrangian for the disk using $x$ as the independent coordinate. Find the equation of motion and solve for the acceleration $\ddot{x}$.

In the following, do not eliminate the constraint of rolling without sliding and use the method of Lagrange multipliers to solve the problem.

(b) (30 points) Write the Lagrangian in terms of $x$ and $\theta$ when considered as two independent coordinates and write the two equations of motion in the presence of the rolling without sliding constraint.

(c) (20 points) Solve the two equations in part (b) together with the constraint equation to find $\ddot{x}$.

(d) (30 points) Find the static friction force between the disk and the incline plane using the method of Lagrange multipliers.

## Problem 2: Classical Mechanics II

In reference frame $S$, a relativistic particle of rest mass $m$ travels with velocity $\vec{v}_1 = \frac{3}{5}c\hat{x}$ in the positive $x$ direction, where $c$ is the speed of light, and hits a stationary particle of identical mass $m$ sitting at the origin, $\vec{v}_2 = 0$. The two particles annihilate and produce a single particle of rest mass $M_f$ that travels with velocity $V_f\hat{x}$.

Express all answers below in terms of $m$ and $c$.

(a) (20 points) What are the momentum 4-vectors of the two particles of mass $m$ before the collision?

(b) (30 points) Assuming that the annihilation process conserves 4-momentum, what are $M_f$ and $V_f$?

Now consider the reference frame $S'$ which is moving with relative velocity $V_f$ in the positive $x$-direction, corresponding to the frame in which the particle of mass $M_f$ sits at rest at the origin.

(c) (25 points) What are the velocities $\vec{v}_1'$ and $\vec{v}_2'$ of the two initial particles in the frame $S'$? (Hint: Use 4-momentum conservation in the frame $S'$.)

(d) (25 points) Some time after the collision the particle of mass $M_f$ subsequently decays to two photons $\gamma_\pm$ which travel back-to-back in the $\pm\hat{x}$ directions. What is the ratio of their energies $E^{'}_{\gamma_+}/E^{'}_{\gamma_-}$ in the $S'$ frame? What is the ratio of their energies $E_{\gamma+}/E_{\gamma-}$ in the $S$ frame?

**QUALIFYING EXAMINATION, Part 2**

**2:30 – 5:00 pm, Thursday September 2, 2021**

**Attempt all parts of both problems.**

Please begin your answer to each problem on a separate sheet, write your 3 digit code and the problem number on each sheet, and then number and staple together the sheets for each problem. Each problem is worth 100 points; partial credit will be given.

Calculators and cell phones may NOT be used.

## Problem 1: Electromagnetism I

A spherical bubble of radius $a$ and conductivity $\sigma_1$ is embedded inside an infinite conducting medium with conductivity $\sigma_2$. A uniform steady current density $\mathbf{J}_0 = J_0\hat{z}$ flows in from infinity. The goal of this problem is to find the distribution of currents everywhere in space given that $\mathbf{J}(|\mathbf{x}| \to \infty) = \mathbf{J}_0$. The relevant equations are the conservation of current $\nabla \cdot \mathbf{J} = 0$, Ohm's law $\mathbf{J} = \sigma\mathbf{E}$, and $\nabla \times \mathbf{E} = 0$.

(a) (15 points) Write down the boundary conditions on the current at the interface between the bubble of conductivity $\sigma_1$ and the medium of conductivity $\sigma_2$.

(b) (15 points) Introduce an electrostatic potential, $\mathbf{E} = -\nabla\Phi$ and use the result of part (a) to write boundary conditions for $\Phi$ at the interface.

(c) (35 points) Use the boundary conditions of part (b) and the condition $\mathbf{J}(|\mathbf{x}| \to \infty) = \mathbf{J}_0$ to solve for the potential $\Phi$ inside and outside the sphere. One possible approach is to use separation of variables in spherical coordinates $(r, \theta, \phi)$ (with $r = 0$ at the center of the sphere). Recall that in spherical coordinates, the general solution of Laplace's equation takes the form

$$\Phi(\mathbf{x}) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right) P_\ell(\cos\theta),$$

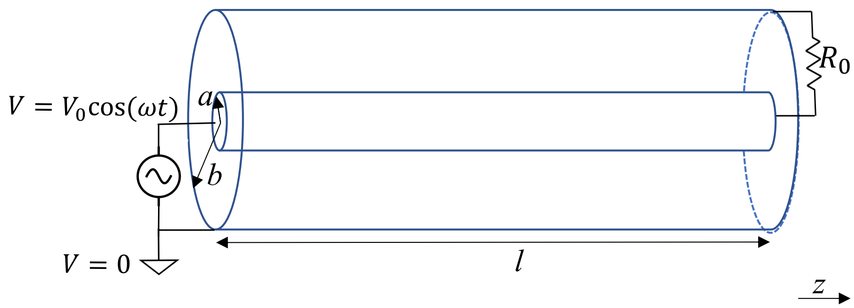
for systems with azimuthal symmetry.

(d) (20 points) Without doing an explicit calculation, explain how would you use your results in (c) to find the surface charge density that develops at the interface between the two conductors.

(e) (15 points) How would you calculate the Ohmic heat produced by the current flow inside the sphere? Do not carry out an explicit calculation.

## Problem 2: Electromagnetism II

A coaxial cable consists of two concentric hollow cylindrical shells separated by vacuum, with radius $a$ for the inner conductor and radius $b$ for the outer conductor (both cylinders have negligible thickness). The length of the conductors, $l$, is much larger than their radial dimensions (i.e., $l \gg a, b$) such that they can be treated as essentially infinite in extent.




A time varying voltage $V(t) = V_0 \cos(\omega t)$ is applied at one end of the cable to the center conductor (at $z = 0$) while the outer shield is grounded. A resistive termination with total resistance $R_0$ is connected at the other end (at $z = l$). Both the voltage source and termination resistor can be assumed to be cylindrically symmetric. Also assume the cylinders can be treated as perfect conductors with negligible resistivity.

For this problem you can work in the "quasistatic limit," i.e., retardation effects can be neglected.

(a) (15 points) Write down a condition on the angular frequency, $\omega$, in terms of $l$ and fundamental constants that should be satisfied to ensure that the quasistatic limit applies.

(b) (30 points) Find the magnetic field everywhere, including terms up to first order in $\omega$ (higher order terms in $\omega$ can be neglected).

(c) (30 points) Find the electric field everywhere, including terms up to first order in $\omega$ (higher order terms in $\omega$ can be neglected).

(d) (25 points) Calculate the energy flux through a cross-section of the cable that is perpendicular to the axis of the cylinder at some position $z$ along its length. Use this to determine the power (energy per unit time) delivered to the resistive termination, $R_0$.

The following expression for the Poynting vector (in SI units) may be useful:

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$$

**QUALIFYING EXAMINATION, Part 3**

**8:30 am – 11:00 pm, Friday September 3, 2021**

**Attempt all parts of both problems.**

Please begin your answer to each problem on a separate sheet, write your 3 digit code and the problem number on each sheet, and then number and staple together the sheets for each problem. Each problem is worth 100 points; partial credit will be given.

Calculators and cell phones may NOT be used.

## Problem 1: Statistical Mechanics I

Certain proteins can sense temperature with surprisingly high sensitivity. While the molecular mechanisms are not well understood, here we will examine a candidate mechanism for a 'molecular thermometer' that reads out temperature by unfolding near a temperature $T_u$. In this problem we will treat unfolding as the readout of the thermometer. (In cells, unfolding would trigger the opening of an ion channel, allowing current to cross the membrane.)

Imagine a polymer, made of $N$ monomers, which can be either 'folded' or 'unfolded'. In the 'folded' state, every monomer's orientation is fixed, and the system has a total energy $E_f = -N\epsilon$ where $\epsilon$ is an energy per monomer associated with folding. In this state, we will assume that the polymer has no configurational entropy. In the 'unfolded' states, each monomer can point in any of 6 directions (choosing an arbitrary direction on a cubic lattice in this model), and we will assume that it has energy $E_u = 0$.

(a) (20 points) What is the configurational entropy of the unfolded state? You do not need to worry about the chain avoiding itself, and do not subtract the folded state from your count.

(b) (20 points) What is the partition function for a single polymer at temperature $T$, assuming the protein can explore both the folded and unfolded states? Note that in this model the entire polymer must be either folded or unfolded – individual monomers cannot independently unbind.

(c) (10 points) At what temperatures $T_{0.5}$ and $T_{0.1}$ are the polymer folded 50% of the time and 10% of the time, respectively?

(d) (10 points) How does $N$ influence the location of the unfolding transition and its sharpness? You can use your result from (c) but also give a qualitative explanation.

(e) (20 points) Denoting the probability of being unfolded by $P_u$, find an expression for the relative sensitivity $g = T\frac{\partial P_u}{\partial T}$ of this thermometer in terms of $T, N$ and $\epsilon$. What is $g$ at the temperature $T_{0.5}$ where 50% of the molecules are unfolded?

(f) (20 points) The most sensitive single molecule thermal sensors in biology can be activated (assume from 10% to 90% unfolded in this model) by about a 3 K change at temperatures near 300 K. If the mechanism through which this is achieved is indeed by unfolding the same molecule, about how long do you expect the polymer to be?

## Problem 2: Statistical Mechanics II

In this problem, we will study the Joule-Thomson (JT) effect in ideal classical and quantum gases. The JT (also known as throttling) effect is the temperature change of a gas or liquid as its pressure changes when it is forced through a valve or porous plug while not allowing heat to be exchanged with the environment. The JT is a process in which the enthalpy remains constant.

Consider a gas of indistinguishable monoatomic bosons of mass $m$ confined to a 3D cubic box of volume $V$. Please look first at the formulas in the next page.

(a) (15 points) Consider first the classical (high-temperature) limit. Recall the equations of state $P(T, V, N)$ and $E(T, V, N)$ of a classical monoatomic ideal gas, and calculate the enthalpy $H(T, V, N)$ of the gas. Find the JT coefficient defined by $\mu_{\rm JT} = \left(\frac{\partial T}{\partial P}\right)_H$ for the classical ideal gas.

(b) (15 points) Show that the chemical potential $\mu$ must satisfy $\mu \leq 0$ in the limit of an infinitely large box. Show that, at a fixed temperature $T$, the density of particles in the excited single-particle states is maximal when $\mu = 0$ and is given by $\rho_c = N_c/V = \zeta(\frac{3}{2})/\lambda_T^3$. Note that the number of particles in the excited single-particle states (i.e., not in the single-particle ground state) is given by $N_{\rm ex} = \int_0^\infty {\rm d}\epsilon\, \mathcal{D}(\epsilon) n(\epsilon)$.

(c) (15 points) What occurs if the density of the gas is further increased above $\rho_c$ at that fixed temperature $T$? Which energy level(s) do the additional atoms occupy?

(d) (15 points) Express the internal energy of the gas $E(\mu, T)$ in the Bose-condensed phase (for which $\mu = 0$) in term of an integral (do not attempt to calculate it). Explain that in the limit of a large box, only the atoms in the excited single-particle states contribute to the energy, i.e., $E \propto N_{\rm ex} k_{\rm B} T = N_c k_{\rm B} T$. Do not attempt to calculate the pre-factor.

(e) (20 points) Instead of the conventional JT process, consider a process by which atoms are removed indiscriminately from the box, i.e., in such a way that the energy per particle $E/N$ remains constant. Show that for the classical ideal gas, this process is also a JT process (known as a JT rarefaction), i.e., that the enthalpy per particle $h = H/N$ is constant. (We note that this result also holds for the ideal Bose gas.)

(f) (20 points) Consider an ideal Bose gas. Using the condition that $E/N$ is constant through the JT rarefaction, show that in the Bose-condensed phase $NT^\gamma$ is constant through the JT process, and determine the value of $\gamma$. Is the JT rarefaction a cooling or heating process? Does the phase-space density $\rho\lambda_T^3$ increase or decrease? ($\rho = N/V$ is the total density).

Useful formulas:

The Bose-Einstein occupation $n(\epsilon)$ of the energy level $\epsilon$ is

$$n(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} - 1},$$

where $\beta = 1/k_{\rm B}T$ and $\mu$ is the chemical potential.

$$\int_0^\infty \frac{x^{1/2}}{e^x - 1} = \frac{\sqrt{\pi}}{2}\zeta\left(\frac{3}{2}\right),$$

where $\zeta$ is Riemann's zeta function.

The single-particle density of states in a 3D box is $\mathcal{D}(\epsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\epsilon}$.

The enthalpy is $H = E + PV$.

The de Broglie thermal wavelength is $\lambda_T = \sqrt{\frac{2\pi\hbar^2}{mk_{\rm B}T}}$.

**QUALIFYING EXAMINATION, Part 4**

**2:30 – 5:00 pm , Friday September 3, 2021**

**Attempt all parts of both problems.**

Please begin your answer to each problem on a separate sheet, write your 3 digit code and the problem number on each sheet, and then number and staple together the sheets for each problem. Each problem is worth 100 points; partial credit will be given.

Calculators and cell phones may NOT be used.

## Problem 1: Quantum Mechanics I

Two *non-interacting* particles are in a 1D box of size $L$ symmetric about $x = 0$:

$$V_{\text{box}}(x) = \begin{cases} 0 & |x| < \frac{L}{2} \\ \infty & \text{otherwise} \end{cases}$$

The two lowest energy (normalized) single-particle eigenstates are

$$\phi_1(x) = \sqrt{\frac{2}{L}} \cos \frac{\pi x}{L} \qquad\qquad \phi_2(x) = \sqrt{\frac{2}{L}} \sin \frac{2\pi x}{L} \ .$$

(a) (10 points) Sketch these eigenfunctions and comment on their parity.

(b) (15 points) Write down the normalized two-particle wavefunction $\psi(x_1, x_2)$ of appropriate symmetry when there is an identical boson in each of these states.

(c) (15 points) Same as (b) but when there is an identical fermion in each of these states.

(d) (20 points) What is $P_{LL}$, the probability of finding both particles in the left half of the box, for the bosonic case?

(e) (20 points) Same as (d) but for the fermionic case.

For parts (d) and (e) do not write down explicit forms of the wave functions, just write them in terms of $\phi_1(x_1)$ etc. and remember their symmetries.

(f) (20 points) What is $P_{LR} + P_{RL}$, the probability of finding the particles in opposite halves of the box (for both the bosonic and fermionic cases)?

Use symmetries to evaluate all of the integrals except $\int_0^{\frac{L}{2}} \phi_1(x)\phi_2(x)dx = \frac{4}{3\pi}$. Write your final answers for $P_{LL}$ amd $P_{LR} + P_{RL}$ in terms of $\frac{4}{3\pi}$. There is no need to calculate numerical values.

## Problem 2: Quantum Mechanics II

*We set $\hbar = 1$ throughout this problem.*

In two-dimensional quantum materials there exist emergent particles which are neither bosons or fermions. In this problem we study quantum mechanics of two identical particles with "semionic" statistics: when they are exchanged, there is a factor of $\pm i$ instead of $\pm 1$ for bosons/fermions. The Hamiltonian operator of the two particles takes the following form:

$$\hat{H} = \frac{\hat{\mathbf{p}}_1^2}{2m} + \frac{\hat{\mathbf{p}}_2^2}{2m} + V(\hat{\mathbf{r}}_1 - \hat{\mathbf{r}}_2).$$

Here $\hat{\mathbf{r}}_1$ and $\hat{\mathbf{r}}_2$ are the coordinates of the two particles in the two-dimensional plane and $\hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2$ are the conjugate momenta. We assume that they have the same mass. It is most convenient to work with center of mass coordinate $\hat{\mathbf{R}} = \frac{\hat{\mathbf{r}}_1 + \hat{\mathbf{r}}_2}{2}$ and relative coordinate $\hat{\mathbf{r}} = \hat{\mathbf{r}}_1 - \hat{\mathbf{r}}_2$. The corresponding momenta are $\hat{\mathbf{P}} = \hat{\mathbf{p}}_1 + \hat{\mathbf{p}}_2, \hat{\mathbf{p}} = \frac{\hat{\mathbf{p}}_1 - \hat{\mathbf{p}}_2}{2}$. We denote the two-particle wavefunction by $\psi(\mathbf{R}, \mathbf{r})$. The semionic statistics is taken care of by imposing the following boundary condition of the two-particle wavefunction:

$$\psi(\mathbf{R}, r, \phi + 2\pi) = -\psi(\mathbf{R}, r, \phi)\ , \tag{1}$$

where $r, \phi$ are the polar coordinates for the relative position $\mathbf{r}$.

(a) (20 points) Rewrite $\hat{H}$ in terms of $\hat{\mathbf{R}}, \hat{\mathbf{P}}, \hat{\mathbf{r}}, \hat{\mathbf{p}}$. Show that the center of mass degree of freedom decouples from the relative one.

*In the following we set $\mathbf{P} = 0$ throughout.*

(b) (20 points) Suppose that $V(\mathbf{r}_1 - \mathbf{r}_2) = V_0(r)$ is a function of $r = |\mathbf{r}_1 - \mathbf{r}_2|$ only. In this case the Hamiltonian is rotationally invariant, so that the angular momentum $\hat{l} = -i\frac{\partial}{\partial \phi}$ is a good quantum number. What are the eigenvalues of $\hat{l}$ given the boundary condition (1)?

(c) (10 points) Since $\hat{l}$ and $\hat{H}$ can be diagonalized simutaneously, we can fix $l$ (to be one of the eigenvalues you find in part (b)) and the Hamiltonian in polar coordinates takes the following form:

$$\hat{H}_l = \frac{1}{m}\left(-\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{l^2}{r^2}\right) + V_0(r).$$

We assume that for each $l$, $\hat{H}_l$ yields a non-degenerate spectrum $E_{ln}$ $(n = 0, 1, 2, \dots)$ of bound states. The radial eigenfunction of $\hat{H}_l$ for the $n$-th bound state will be denoted by $f_{ln}(r)$, so $\hat{H}_l f_{ln}(r) = E_{ln} f_{ln}(r)$. You may assume that $f_{ln}(r)$ are *real* and properly normalized.

Show that the spectrum of $\hat{H}$ (with $\mathbf{P} = 0$) is two-fold degenerate.

(d) (10 points) Besides rotational symmetry, the Hamiltonian is invariant under the time-reversal transformation $\hat{\Theta}$, which is just complex conjugation of the wavefunction in coordinate space. How do the eigenstates transform under $\hat{\Theta}$? Your result should satisfy $\hat{\Theta}^2 = 1$.

(e) (10 points) The Hamiltonian is also invariant under the rotation $\phi \to \phi + \pi$ (which amounts to an exchange of the two particles $\mathbf{r}_1 \leftrightarrow \mathbf{r}_2$). We will call this operation $\hat{C}$. Define $\hat{\Theta}' = \hat{\Theta}\hat{C}$, which is another anti-unitary symmetry transformation. Determine how the eigenstates transform under $\hat{\Theta}'$.

(f) (10 points) Show that $\hat{\Theta}'^2 = -1$, so the degenerate eigenstates in fact form Kramers' doublets under the modified time-reversal symmetry $\hat{\Theta}'$.

(g) (20 points) Now suppose that the two-particle interaction has a small anisotropic component, so $V(\mathbf{r}) = V_0(r)(1 + \epsilon \cos^2 \phi)$ with $\epsilon \ll 1$. Use perturbtion theory to find the first-order corrections (i.e., proportional to $\epsilon$) to energy eigenvalues. Your result may contain an integral involving $f_{ln}$ and $V_0$. Does the degeneracy of the spectrum persist once the perturbation is turned on? Explain why.