QUALIFYING EXAMINATION, Part 1

9:00 – 11:30 am, Monday August 29, 2022

Attempt all parts of both problems

Please begin your answer to each problem on a separate sheet, write your 3 digit code and the problem number on each sheet, and then number and staple together the sheets for each problem. Each problem is worth 100 points; partial credit will be given.

Calculators and cell phones may NOT be used.

Problem 1: Classical Mechanics I

An asteroid of mass m and initial velocity v is traveling towards a distant planet with mass M and radius R. Assume $m \ll M$ and that the planet is perfectly spherical and has no atmosphere.

(a) (10 points) Find the angular momentum l of the asteroid relative to the center of the planet assuming the asteroid initially approaches the planet with an impact parameter b. Express l as a function of m, v and b.

(b) (20 points) For this central force problem, the energy can be written in the form

$$E = \frac{1}{2}\mu\dot{r}^2 + V_{eff}(r) \ ,$$

where the effective potential $V_{eff} = V(r) + l^2/(2\mu r^2)$ for the central potential $V(r)$ (in this case gravity). Find the reduced mass, μ, in the limit $m \ll M$ as above. Use this and the result from (a) to find the effective potential $V_{eff}(r)$ in terms of M, m, v, b and the gravitational constant G.

(c) (30 points) Find the range of values for b for which the asteroid will hit the planet.

Hint: The asteroid will just hit the planet when the distance of closest approach is R. Use the appropriate conservation law and that $\dot{r} = 0$ at the distance of closest approach.

In the following parts we will find an expression for the scattering angle at the impact parameter at which the asteroid just barely misses the planet.

(d) (20 points) In general, the scattering angle can be written in terms of the central potential, $V(r)$, as:

$$\theta = \pi - 2\int_{r_m}^{\infty} \frac{b\,dr}{r\sqrt{r^2\left(1 - \frac{V(r)}{E}\right) - b^2}} = \pi - 2\int_0^{u_m} \frac{b\,du}{\sqrt{1 - \frac{V(u)}{E} - b^2u^2}} \ ,$$

where r_m is the distance of closest approach, and we have made the convenient transformation $u = 1/r$ in the second expression.

Use the expression above to find the scattering angle as a function of the impact parameter b. The following integral may be useful:

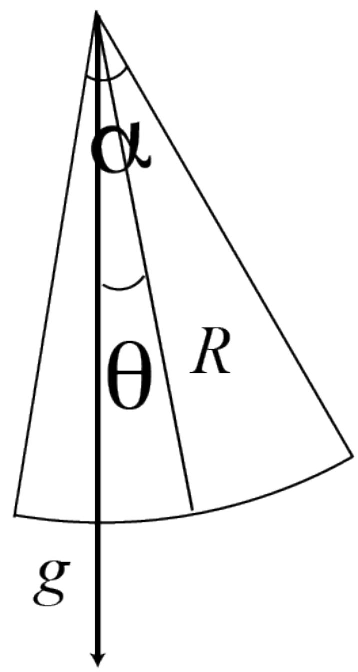
$$\int_0^{x_0} \frac{dx}{\sqrt{1 - \alpha x - x^2}} = \frac{\pi}{2} - \arctan\left(\frac{|\alpha|}{2}\right)$$

for x_0 defined as the positive solution of $1 - \alpha x_0 - x_0^2 = 0$.

(e) (20 points) Evaluate the scattering angle formula above for the value of b such that the asteroid just barely misses hitting the planet. Check that the limits for $v \to \infty$ and $v \to 0$ make sense.

Problem 2: Classical Mechanics II

A section of a flat uniform disk of radius R, apex angle α, and mass m is suspended from the top as a compound pendulum under the influence of gravity g. We denote by θ the angle of displacement from the vertical. We assume that this system is released from rest at an initial displacement angle of θ_0.



(a) (20 points) Find the moment of inertia of this compound pendulum.

(b) (10 points) Find its kinetic energy in terms of $\dot{\theta}$.

(c) (20 points) Find its potential energy as a function of θ.

(d) (20 points) Using the Lagrangian formalism, and the results of parts (b) and (c), find the equation of motion for θ. Calculate the angular frequency ω_0 in the limit of small oscillations.

(e) (10 points) Compare your result for ω_0 in part (d) as a function of α to the angular frequency of a simple pendulum where all of the mass is concentrated at a distance R from the point of support. For which value of α are their frequencies equal?

(f) (20 points) Find the equation for $\dot{\theta}$. Use this equation to obtain an expression for the period T of the motion as an integral over θ for an arbitrary initial angle θ_0.
Hint: Use energy conservation or the equation of motion for θ.

QUALIFYING EXAMINATION, Part 2

1:00 – 3:30 pm, Monday August 29, 2022

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Problem 1: Electromagnetism I

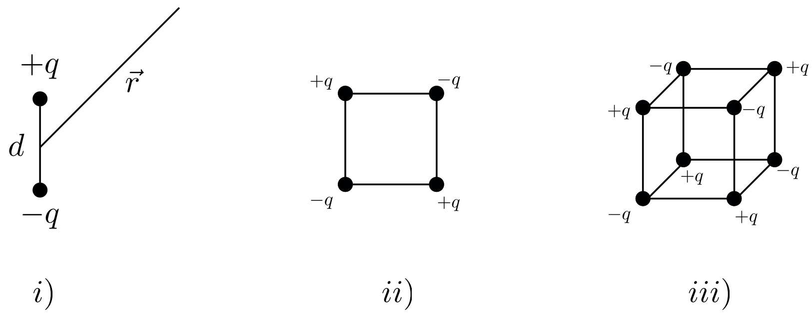

Figure 1: Charge configurations (i), (ii) and (iii).

(a) (30 points) Compute the exact potential $V(\vec{r})$ for the configuration of charges illustrated in Fig 1 (i). Expand the result in $d/r \ll 1$, keeping only the leading nonzero term in the expansion. How does this potential fall off as a function of r?

(b) (15 points) Re-compute your result in (a) by determining the dipole moment

$$\vec{p} = \int \vec{r}\,'\rho(\vec{r}\,')d\tau'$$

of the charge configuration in (i), and using

$$V_{\text{dip}}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}\,.$$

(c) (10 points) What are the dipole moments of the charge configurations in (ii) and (iii)?

(d) (30 points) The quadrupole moment of a charge distribution is defined as

$$Q_{ij} = \int \left(3r'_i r'_j - (r')^2\delta_{ij}\right)\rho(\vec{r'})d\tau'\,,$$

and generates a potential

$$V_{\text{quad}}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{2r^3}\sum_{i,j}\hat{r}_i\hat{r}_j Q_{ij}\,.$$

Here r'_i is the i-th component of the vector $\vec{r}\,'$ and $\hat{r}_i = r_i/r$.

Compute the angular dependence (in general this could be a function of both angles θ, φ) of the potential $V_{\text{quad}}(\vec{r})$ for the charge configuration in Figure 1 (ii), centered at the origin. Assume configuration (ii) is in the $x-y$ plane and all sides are of length d.

(e) (15 points) Using symmetry arguments or through direct application of the formula for the quadrupole moment, compute the quadrupole moment for the charge configuration in (iii) around its center. With which power of r will the potential for this configuration fall off as?

Problem 2: Electromagnetism II

An infinitely long wire with linear charge density $-\lambda$ runs along the z-axis and through the origin. An infinitely long insulating cylindrical shell entirely located at radius R, with moment of inertia I *per unit length* is concentric with the wire. The cylinder is free to rotate frictionlessly about the z-axis and it carries a uniform surface charge density $\sigma = \frac{\lambda}{2\pi R}$. The system exists inside a region of externally applied uniform magnetic field along the z direction, $\vec{B} = B_{ex}\hat{z}$.

Starting at $t = 0$ the magnetic field is slowly reduced to zero over a time $T \gg R/c$, where c is the speed of light. Eventually, when the external magnetic field is reduced to zero, the cylinder has a final angular velocity ω. You can assume that the final angular velocity is sufficiently slow such that relativistic effects can be ignored.

(a) (25 points) What is the magnetic field inside the cylinder at $t \geq T$?
Hint: the rotating charged cylinder generates a current of magnitude $\sigma\omega R$ per unit length.

(b) (25 points) The initial angular momentum is entirely stored in the fields. Compute the initial angular momentum pointing along the z axis per unit length.

(c) (25 points) This initial angular momentum is partially converted to mechanical angular momentum over time, causing the cylinder to rotate. Compute the final *total* angular momentum pointing along the z axis per unit length at $t \geq T$ in terms of λ, R, I, and ω.

(d) (25 points) Use conservation of total angular momentum to solve for the final angular velocity ω. If the initial external magnetic field B_{ex} is large, does it spin faster or slower than if B_{ex} is small?

Useful formulas (SI units)

- Gauss's law:
$$\int \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{enc}$$
- Ampere's law:
$$\int \vec{B} \cdot \vec{dl} = \mu_0 I_{enc}\ .$$
- The angular momentum density $\vec{l}_{EM}$ stored in an electromagnetic field is
$$\vec{l}_{EM} = \vec{r} \times \vec{p}_{EM} = \epsilon_0 \vec{r} \times (\vec{E} \times \vec{B})$$
where $\vec{p}_{EM} = \epsilon_0 \vec{E} \times \vec{B}$ is the linear momentum density.
-
$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\, \vec{B} - (\vec{A} \cdot \vec{B})\, \vec{C}$$

QUALIFYING EXAMINATION, Part 3

9:00 – 11:30 am, Tuesday August 30, 2022

Attempt all parts of both problems.

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Problem 1: Quantum Mechanics I

The deuteron is a bound state of a proton and a neutron. It may be described crudely by a spherical square well of depth $V_0 > 0$ and range a. The radial Schrödinger equation is

$$-\frac{\hbar^2}{2\mu r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{\hbar^2 l(l+1)}{2\mu r^2}R=(E+V_0)R \qquad \text{for } r<a\,,$$

and

$$-\frac{\hbar^2}{2\mu r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{\hbar^2 l(l+1)}{2\mu r^2}R=ER \qquad \text{for } r>a\,,$$

where E is a bound state energy with $-V_0 \leq E \leq 0$ and where μ is the reduced mass of the deuteron (one half the nucleon mass). There is a single s-wave bound state with $E = -2.226$ MeV.

(a) (20 points) By substituting $R(r) = u(r)/r$, determine the form of the physically allowed (normalizable) s-wave solution $u(r)$ inside the well.

(b) (20 points) Do the same as in (a) but outside the well.

(c) (20 points) By matching the logarithmic derivatives of $u(r)$ at the boundary $r = a$, write down an equation for the energy eigenvalues.
(Note: the logarithmic derivative of a function $f(x)$ is defined by $d\ln f(x)/dx = f'(x)/f(x)$.)

(d) (20 points) The bound state has $|E| \ll V_0$. Solve the energy eigenvalue equation in the approximation $E \to 0$ to determine the value of the dimensionless quantity $\sqrt{2\mu V_0/\hbar^2}\,a$.

(e) (20 points) Taking $a \simeq 10^{-15}$ meter and using the fact that $\hbar^2\pi^2/8\mu \simeq 10^{-28}$ MeV-meters^2, estimate V_0. Compare it to $|E|$.

Problem 2: Quantum Mechanics II

We set $\hbar = 1$ throughout the problem.

A charge-neutral spin-$\frac{1}{2}$ particle can couple to an external electric field $\mathbf{E}$ through a spin-orbit (SO) interaction:

$$\hat{H}_{\text{SO}} = \gamma \hat{\mathbf{s}} \cdot (\hat{\mathbf{p}} \times \mathbf{E}) \ ,$$

where $\mathbf{s}$ is the spin operator, $\mathbf{p}$ is the linear momentum, and γ is a coupling constant.

Here we study the effect of this interaction on the spectrum of a neutral particle moving in a harmonic potential. Thus the Hamiltonian of the particle takes the form:

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{SO}}, \qquad \hat{H}_0 = \frac{\hat{\mathbf{p}}^2}{2m} + \frac{1}{2} m\omega^2 \hat{\mathbf{r}}^2.$$

$\hat{H}_0$ is equivalent to 3 independent harmonic oscillators in the x, y and z directions, so its eigenstates can be labeled by $|n_x, n_y, n_z, s_z = \pm\frac{1}{2}\rangle$, where $n_{x/y/z} = 0, 1, \dots$ are the occupation numbers. Its energy levels are given by $E_{n_x,n_y,n_z,s_z} = \omega N + \frac{3}{2}$, where $N = n_x + n_y + n_z$.

The spin degree of freedom is not involved in (a) and (b) below, and will be omitted.

(a) (20 pts) Consider the $N = 0$ state without the SO interaction. What is its total orbital angular momentum? Explain your answer.

(b) (30 pts) When $\gamma = 0$, the three eigenstates with $N = 1$ form the spin-1 multiplet of the orbital angular momentum $\mathbf{l}$. The $\hat{l}_z$ operator can be written as (no need to prove it):

$$\hat{l}_z = i(\hat{a}_y^\dagger \hat{a}_x - \hat{a}_x^\dagger \hat{a}_y) \ ,$$

where $\hat{a}_{x/y/z}$ are the lowering operators for each of the oscillators (see **useful formulas** below). Find the angular momentum basis $|N = 1, l = 1, m\rangle$, where m is the $\hat{l}_z$ eigenvalue, in terms of $|n_x, n_y, n_z\rangle$.

Below we denote the eigenstates of $\hat{H}_0$ in the angular momentum basis by $|N, l, m, s_z\rangle$.

(c) (20 pts) Consider the parity transformation:

$$\hat{I} : \hat{\mathbf{r}} \to -\hat{\mathbf{r}}, \ \ \hat{\mathbf{p}} \to -\hat{\mathbf{p}}, \ \ \hat{\mathbf{s}} \to \hat{\mathbf{s}},$$

which satisfies $\hat{I}^2 = 1$. As a matter of fact we know that $|N, l, m, s_z\rangle$ are all eigenstates of $\hat{I}$: $\hat{I}\,|N, l, m, s_z\rangle = (-1)^l\,|N, l, m, s_z\rangle$. Use this to show that $\langle N, l, m, s_z|\hat{H}_{\text{SO}}|N, l, m', s_z'\rangle = 0$.

(d) (20 pts) Treating $\hat{H}_{\text{SO}}$ as a perturbation, show that both the $N = 0$ and $N = 1$ levels remain degenerate to first order in γ.

(e) (10 pts) The degeneracy of the $N = 0$ states in fact remains robust to all orders in perturbation theory. Give a symmetry argument to explain this degeneracy.
Hint: Consider time-reversal symmetry.

Useful formulas

- The lowering operator for a quantum oscillator along the x direction is defined as

$$\hat{a}_x = \sqrt{\frac{m\omega}{2}}\Big(\hat{x} + \frac{i}{m\omega}\hat{p}_x\Big).$$

  Inverting this expression we have

$$\hat{x} = \sqrt{\frac{1}{2m\omega}}(\hat{a}_x^\dagger + \hat{a}_x), \quad \hat{p}_x = i\sqrt{\frac{m\omega}{2}}(\hat{a}_x^\dagger - \hat{a}_x).$$

- The occupation number operator is $\hat{n}_x = \hat{a}_x^\dagger \hat{a}_x$. The matrix elements of raising/lowering operators on occupation number eigenstates are

$$\hat{a}_x^\dagger \left|n_x\right\rangle = \sqrt{n_x + 1}\left|n_x + 1\right\rangle, \quad \hat{a}_x \left|n_x\right\rangle = \sqrt{n_x}\left|n_x - 1\right\rangle.$$

  Similar formulas can be written for the oscillators in the y and z directions.

QUALIFYING EXAMINATION, Part 4

1:00 – 3:30 pm, Tuesday August 30, 2022

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Problem 1: Statistical Mechanics I

The canonical partition function of a system N particles in contact with a heat reservoir at temperature T is defined as

$$Z(\beta) = \sum_i e^{-\beta E_i} ,$$

where $\beta = 1/(kT)$ and i labels the energy eigenstates of the N-particle system.

In the following we consider a system with 3 single-particle energy levels $0, \varepsilon, 2\varepsilon$.

(a) (10 points) Indicate the ways to populate the 3 levels with 2 identical fermions. Show each level as a horizontal line with the level 0 at the bottom and put an x for each particle in that level.

(b) (10 points) Repeat (a) but for 2 identical bosons.

(c) (20 points) Write down the partition function $Z(\beta)$ for the case of 2 non-interacting fermions and compute the mean energy $\langle E \rangle$. Relate the mean energy to a certain derivative of Z.

(d) (20 points) Write down the partition function $Z(\beta)$ for the case of 2 non-interacting bosons and compute the mean energy $\langle E \rangle$.

The grand-canonical partition function $Z(\mu, \beta)$ for a system in equilibrium with a particle and heat reservoir with chemical potential μ and temperature T is defined as

$$Z(\mu, \beta) = \sum_N \sum_i e^{\beta(\mu N - E_{i,N})} , \tag{1}$$

where N is the number of particles and $E_{i,N}$ is the energy of any allowed state i of the N-particle system.

(e) (20 points) Go back to the 3 level system above. Use Eq. (1) to write explicitly the sum over N (in the order of increasing N), and at each N, the sum over $E_{i,N}$ for a system of non-interacting fermions.

Calculate $Z(\mu, \beta)$ by treating each *level* as an independent system of fermions in contact with a particle and heat reservoir at some fixed μ and β, and show that the expression you find is equivalent to the explicit formula you have written using Eq. (1).

(f) (20 points) Compute $Z(\mu, \beta)$ for non-interacting bosons by treating each *level* as an independent system of non-interacting bosons in contact with particle and heat reservoir at some fixed μ and β.

Problem 2: Statistical Mechanics II

Consider a mixture of noninteracting (spinless) fermions and bosons confined in a 3D cubic box of length L. The fermion and boson have equal mass m. The mixture is in equilibrium at a temperature T.

(a) (10 points) Consider first a single particle of mass m. Show that the single-particle density of states is

$$\mathcal{D}(\epsilon) = C\sqrt{\epsilon},$$

where the dimensionful constant $C = \frac{L^3}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}$. Use the boundary conditions that the single-particle wave function vanishes at the sides of the cubic box.

For ease of calculation, we will work in the following in the grand-canonical ensemble. We will assume that the bosons and fermions are connected to *independent* particle reservoirs.

(b) (15 points) In the grand-canonical ensemble, what are the independent thermodynamic variables to describe this mixture in equilibrium? What is the corresponding thermodynamic potential? Express this potential in terms of the total internal energy U, the total entropy S, the average particle numbers N_{B} and N_{F}, and the independent variables.

(c) (10 points) Write the equations for the average particle numbers N_{B} and N_{F} in terms of suitable partial derivatives of the thermodynamic potential in part (b).

(d) (20 points) Since the mixture is noninteracting, how can you express the partition function of the mixture in terms of the partition functions Z_{B} and Z_{F} of its bosonic and fermionic subparts, respectively? Use this to express the average particle numbers N_{B} and N_{F} in term of an integral involving $\mathcal{D}(\epsilon)$ and the independent variables you found in (b).

Hint: Use (c) to express the average particle numbers in terms of suitable partial derivatives of the grand-canonical partition function.

(e) (15 points) Similarly, express the internal energy U of the system in terms of an integral involving $\mathcal{D}(\epsilon)$ and the independent variables you found in (b).

(f) (20 points) Determine U of the mixture at $T = 0$, as a function of N_{B} and N_{F}. What is the corresponding state of the system? What are the chemical potentials of the bosons and of the fermions in that state?

(g) (10 points) Sketch the box trap (in 1D for clarity) with a few single-particle energy levels (not to scale). Represent on this sketch by circles the occupied levels of excited many-particle states of the Bose and Fermi gases that contribute at a very low (but nonzero) temperature.

Useful formulas

- The partition functions Z_B and Z_F in the grand-canonical ensemble for a gas of bosons and of fermions are, respectively

  $$\ln Z_\mathrm{B}(\mu_\mathrm{B}, T) = -\sum_j \ln(1 - e^{-\beta(\epsilon_j - \mu_\mathrm{B})})$$

  and

  $$\ln Z_\mathrm{F}(\mu_\mathrm{F}, T) = \sum_j \ln(1 + e^{-\beta(\epsilon_j - \mu_\mathrm{F})}) \ ,$$

  where the sum is taken over all single-particle states j of energy ϵ_j.

- The grand potential Ω is given in terms of the grand-canonical partition function Z by

  $$\Omega = -kT \ln Z \ .$$