QUALIFYING EXAMINATION, Part 1

9:00 – 11:30 am, Monday August 28, 2023

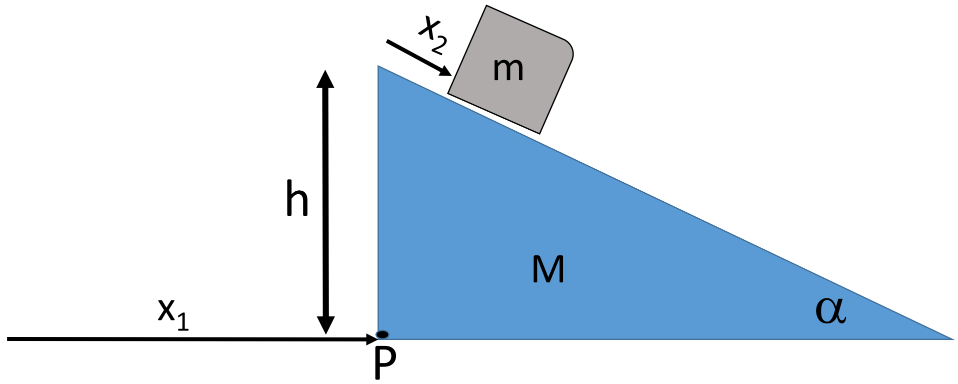
Attempt all parts of both problems.

Please begin your answer to each problem on a separate sheet, write your 3 digit code and the problem number on each sheet, and then number and staple together the sheets for each problem. Each problem is worth 100 points; partial credit will be given.

Calculators and cell phones may NOT be used.

Problem 1: Classical Mechanics I

At $t = 0$, a block of mass m and negligible size is resting at the top of a frictionless inclined plane of mass M, with an incline α and height h, which in turn slides on a horizontal frictionless table.



(a) (25 points) Write down the kinetic energy T and potential energy V in terms of the coordinates x_1 and x_2 and the corresponding velocities. Choose the zero of the potential to be at the location of m at $t = 0$.

(b) (10 points) What are the canonical momenta p_1 and p_2?

(c) (15 points) Write the Euler-Lagrange equations for x_1 and x_2.

(d) (10 points) Identify and interpret the conserved momentum.

(e) (25 points) At what time t^* does m reach the bottom of the incline?

(f) (15 points) What is $x_1(t^*)$, the location at time t^* of the point P at the lower left corner of the plane?

Problem 2: Classical Mechanics II

(a) (20 points) A clock moving at velocity $v = \frac{3}{5}c$ passes me (using unprimed coordinates) sitting at the origin at $t = t' = 0$. What is the clock's location in my frame when $t' = 1s$ *in its frame*?

(b) (25 points) If at this time $t' = 1\,s$ the clock emits a light pulse back to me, at what time t, per my clock, does it reach me?

The parts below are an independent problem.

(c) (25 points) In my frame event B occurs $4\,\mu s$ after and $3\,km$ to the right of event A. What is the velocity v of the frame in which A and B occur simultaneously?

(d) (15 points) Can the temporal order of the events in (c) be reversed in some frame? Explain.

(e) (15 points) Is there a frame in which the events in (c) occur at the same point?

Useful equations

The Lorentz transformation from a frame S to a frame S' moving at a speed v (with respect to S) along the x direction is given by

$$\begin{aligned} x' &= \gamma\,(x - vt)\ , \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right)\ , \end{aligned}$$

where $\gamma = 1/\sqrt{1-(v/c)^2}$, and $c = 3\cdot 10^8\ \frac{m}{s}$ is the speed of light.

QUALIFYING EXAMINATION, Part 2

1:00 – 3:30 pm, Monday August 28, 2023

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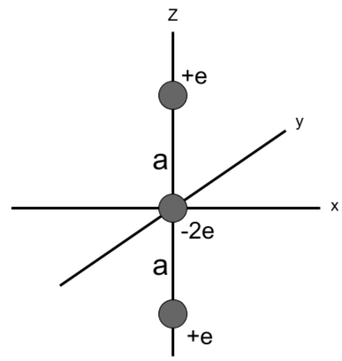
Problem 1: Electromagnetism I

Consider a set of three charges aligned on the $\hat{z}$ axis: +e at $-a\hat{z}$, +e at $+a\hat{z}$, and $-2e$ at the origin as shown in the figure. In this problem we use Gaussian units.



(a) (45 points) Find the non-vanishing terms up to third order in the multipole expansion of the potential $\Phi(\vec{x})$ in spherical coordinates r, θ, and φ. As a reminder the multipole expansion is given by:

$$\Phi(\vec{x}) = \frac{Q}{r} + \frac{\vec{P}\cdot\vec{x}}{r^3} + \frac{1}{2}\frac{Q_{ij}x_i x_j}{r^5} + \ldots$$

where $r = |\vec{x}|$ and

$$Q = \int \rho(\vec{x}')dV' \ , \quad P_i = \int \rho(\vec{x}')x_i' dV'$$

$$Q_{ij} = \int \rho(\vec{x}')(3x_i'x_j' - \delta_{ij}r'^2)dV' \ .$$

(b) (10 points) Express your answer in (a) in terms of Legendre polynomials.

(c) (25 points) Calculate the asymptotic electric field $\vec{E}$ in spherical coordinates.

(d) (20 points) Find the exact expression for the electric field at a point on the positive z axis at a distance $r > a$ from the origin and compare its asymptotic behavior for $r \gg a$ with your results in part (c).

Useful formulas:

The first few Legendre polynomials are given by

$$P_0(x) = 1 \ , \quad P_1(x) = x \ , \quad P_2(x) = \frac{1}{2}(3x^2 - 1) \ .$$

In spherical coordinates

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \ .$$

Problem 2: Electromagnetism II

The pattern of radiation from a confined source can be determined by solving the wave equations that derive from Maxwell's equations. In the Lorentz gauge, each takes the form

$$\left(\vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi(\vec{x},t) = -4\pi\ f(\vec{x},t)$$

where $\psi(\vec{x},t) = \vec{A}(\vec{x},t)$ or $\phi(\vec{x},t)$ and $f(\vec{x},t) = \frac{1}{c}\ \vec{J}(\vec{x},t)$ or $\rho(\vec{x},t)$, respectively. Here $\vec{A}$ and ϕ are the vector and scalar potentials, and $\vec{J}$ and ρ are the current and charge densities. Gaussian units are employed and c is the speed of light. The retarded (causal) solution to each equation is:

$$\psi(\vec{x},t) = \int d^3x' \frac{f(\vec{x}',\ t' = t - \frac{|\vec{x}-\vec{x}'|}{c})}{|\vec{x}-\vec{x}'|} .$$

(a) (20 points) Isolate a single frequency by Fourier transforming these solutions in time. Express $\tilde{\vec{A}}(\vec{x},\omega)$ $(\equiv \frac{1}{2\pi}\int dt\ e^{i\omega t}\vec{A}(\vec{x},t))$ in terms of $\tilde{\vec{J}}(\vec{x}',\omega)$ $(\equiv \frac{1}{2\pi}\int dt\ e^{i\omega t}\vec{J}(\vec{x}',t))$. Do the same relating $\tilde{\phi}(\vec{x},\omega)$ to $\tilde{\rho}(\vec{x}',\omega)$. What is the charge conservation equation relating $\tilde{\vec{J}}(\vec{x}',\omega)$ to $\tilde{\rho}(\vec{x}',\omega)$?

(b) (20 points) The three distance scales are the size d of the source, the wavelength $\lambda = 2\pi c/\omega$, and the distance of observation $r = |\vec{x}|$. Take $\lambda \gg d$ as in the case of the AM radio band, and work in the far (radiation) zone $r \gg \lambda$. What is the leading ("electric-dipole") integral expression for $\tilde{\vec{A}}(\vec{x},\omega)$? How does it fall with r?

(c) (20 points) By employing the fact that $\int d^3x'\ \vec{\nabla}' \cdot \left[x_i'\ \tilde{\vec{J}}(\vec{x}',\omega)\right] = 0$, re-express the leading expression for $\tilde{\vec{A}}(\vec{x},\omega)$ in terms of the Fourier transform of the electric-dipole moment of the charge distribution $\tilde{\vec{p}}(\omega) \equiv \int d^3x'\vec{x}'\tilde{\rho}(\vec{x}',\omega)$.

(d) (20 points) Using this expression, what is the leading form of the Fourier transform of the magnetic field $\tilde{\vec{B}}(\vec{x},\omega)$ in the far zone? What is $\tilde{\vec{E}}(\vec{x},\omega)$ in terms of $\tilde{\vec{B}}(\vec{x},\omega)$? *Hint:* To find $\tilde{\vec{E}}$ use Ampére's law in vacuum.

(e) (20 points) Finally, consider the single-frequency case $\tilde{\rho}\left(\vec{x}',\omega\right) = \rho\left(\vec{x}'\right)\delta\left(\omega-\omega_0\right).$ Fourier transform the leading expressions for $\tilde{\vec{A}}(\vec{x},\omega)$ back to give the time dependent expressions for $\vec{A}(\vec{x},t)$.

QUALIFYING EXAMINATION, Part 3

9:00 – 11:30 am, Tuesday August 29, 2023

Attempt all parts of both problems.

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Problem 1: Quantum Mechanics I

This problem explores the time evolution of a state when its Hamiltonian is time dependent, i.e., $H = H(t)$. This problem amounts to the first few steps that would be taken in deriving the adiabatic theorem and the geometric phase.

The time evolution of the system is governed by the Schrödinger equation (taking $\hbar = 1$):

$$i\,|\dot{\psi}(t)\rangle = H(t)\,|\psi(t)\rangle \ ,$$

where throughout a dot over a symbol denotes a derivative with respect to time.

We denote the instantaneous eigenstates of $H(t)$ by $|n(t)\rangle$ with eigenvalues $E_n(t)$, i.e.,

$$H(t)\,|n(t)\rangle = E_n(t)\,|n(t)\rangle \ . \tag{1}$$

For simplicity we assume that at each time $E_n(t) \neq E_m(t)$ for all $n \neq m$.

We expand $|\psi(t)\rangle$ in this basis

$$|\psi(t)\rangle = \sum_n c_n(t) e^{i\theta_n(t)}\,|n(t)\rangle \ ,$$

where each $c_n(t)$ is a complex number and

$$\theta_n(t) = -\int_0^t E_n(t')dt' \ .$$

Note that the factors $e^{i\theta_n(t)}$ in the above expansion are not strictly necessary, but they do make things simpler.

In the following, note that quantities like $H, |n\rangle, E_n, c_n, \theta_n$, etc., are time-dependent, even though their dependence is not written explicitly.

(a) (25 points) Show that the coefficients $c_n(t)$ obey:

$$\sum_n e^{i\theta_n}\left(\dot{c}_n\,|n\rangle + c_n\,|\dot{n}\rangle\right) = 0 \ .$$

(b) (25 points) Use the result in (a) show that any particular coefficient c_k obeys:

$$\dot{c}_k = -\sum_n c_n e^{i(\theta_n - \theta_k)}\,\langle k|\dot{n}\rangle \ .$$

(c) (25 points) Show that for $k \neq n$, the quantity $\langle k|\dot{n}\rangle$ can be written as

$$\langle k|\dot{n}\rangle = \frac{\langle k|\dot{H}|n\rangle}{E_n - E_k} \ .$$

Hint: Make use of Eq. (1).

(d) (25 points) Show that when $k = n$, the quantity $\langle k|\dot{n}\rangle$ is a purely imaginary number.

Hint: Use the relation $\langle n|n\rangle = 1$.

Problem 2: Quantum Mechanics II

In this problem we will study the spectrum of the hydrogen atom algebraically, following Pauli. The Hamiltonian describing the hydrogen atom is

$$H = \frac{\vec{p}^2}{2m_e} - \frac{e^2}{r} .$$

(a) (10 points) Use symmetry considerations to argue that the angular momentum is conserved, and therefore $[L_i, H] = 0$.

(b) (15 points) An additional conserved quantity for the $1/r$ potential is the Laplace-Runge-Lenz vector, which in quantum mechanics is defined by

$$\vec{A} = \frac{\vec{p} \times \vec{L} - \vec{L} \times \vec{p}}{2m_e} - e^2 \frac{\vec{r}}{r} .$$

Show that $\vec{A}$ is hermitian.

Next, we consider the two linear combinations

$$\vec{T} = \frac{1}{2}\left(\vec{L} + \sqrt{-\frac{m_e}{2E}}\vec{A}\right) , \qquad \vec{S} = \frac{1}{2}\left(\vec{L} - \sqrt{-\frac{m_e}{2E}}\vec{A}\right) ,$$

where $E < 0$ is an energy eigenvalue. It can be shown that $\vec{T}$ and $\vec{S}$ form two commuting copies of the $SU(2)$ algebra familiar from the study of angular momentum and that their squares are equal and given by

$$\vec{T}^2 = \vec{S}^2 = -\frac{m_e e^4}{8E} - \frac{1}{4} , \tag{2}$$

where here and in the following we set $\hbar = 1$.

(c) (20 points) Just like how we quantize angular momentum $\vec{J}$ using $\vec{J}^2$ and J_z, we can label the eigenstates by $|\Psi\rangle = |t, m_T, m_S\rangle$ where

$$T^2|\Psi\rangle = S^2|\Psi\rangle = t(t+1)|\Psi\rangle , \qquad T_z|\Psi\rangle = m_T|\Psi\rangle , \qquad S_z|\Psi\rangle = m_S|\Psi\rangle .$$

What are the allowed values of t, m_T, and m_S?

(d) (20 points) Use the relation (2) to solve for the energy levels E of the hydrogen atom. They may look more familiar if expressed in terms of $n = 2t + 1$.

(e) (15 points) What is the degeneracy of the level labelled by n?

(f) (20 points) For a given n, determine the allowed values l of the orbital angular momentum. Find the degeneracy of the level labelled by n (using the allowed values of l) and compare with your result in (e).

Hint: Use $\vec{L} = \vec{T} + \vec{S}$ and apply the rule for adding the two angular momenta $\vec{T}$ and $\vec{S}$.

QUALIFYING EXAMINATION, Part 4

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Problem 1: Statistical Mechanics I

Consider a large vertical cylinder filled with N distinguishable gas particles of mass m at equilibrium at inverse temperature β. This cylinder has an area A and is closed at the bottom, but has no lid (assume the cylinder extends to infinity). The particles are held by gravity which we take to be constant g, so that the gravitational potential energy of a single particle is mgz, where the z axis is taken along the symmetry axis of the cylinder and $z = 0$ is the bottom of the cylinder. Assume the classical limit and work in the canonical ensemble.

(a) (20 points) Calculate the classical partition function of a single particle. What is the partition function of the N particles?

(b) (20 points) Express the average height $\langle z \rangle$ of a particle in the gas as an appropriate logarithmic derivative of the single-particle partition function. Calculate this average height using the expression you found in part (a) for the single-particle partition function.

(c) (20 points) Find the pressure $P(z)$ as a function of z as you would measure with a barometer.

Hint: Consider the balance of forces acting on a thin layer of the cylinder at height z.

(d) (10 points) What is the total downward force acting on the bottom of the cylinder?

(e) (20 points) Does the pressure at a given height z increase or decrease with temperature?

(f) (10 points) From this analysis, on a colder day, how do you expect the pressure at the surface of the earth to change? What about at high elevation? Ignore the many important details, and in particular that temperature itself is a function of height in the atmosphere.

Problem 2: Statistical Mechanics II

Consider an electron gas at $T = 0$ consisting of $N_\uparrow$ spin up and $N_\downarrow$ spin down electrons in a 3D cubic box of volume V. We will first assume that the electrons are non-interacting.

(a) (10 points) What is the ground state of each of the spin-up and spin-down Fermi gases? Write down the total occupation number (*i.e.*, of both spin states) $\rho_{\rm tot}(\epsilon)$ of a single-particle energy level ϵ as a function of the two Fermi energies $\epsilon_{\rm F\uparrow}$ and $\epsilon_{\rm F\downarrow}$.

(b) (10 points) Show that the Fermi energies are given by $\epsilon_{\rm F\uparrow} = \alpha n_\uparrow^{2/3}$ and $\epsilon_{\rm F\downarrow} = \alpha n_\downarrow^{2/3}$, where $n_\uparrow = N_\uparrow/V$, $n_\downarrow = N_\downarrow/V$ are the densities of the spin up and spin down electrons and $\alpha = \frac{\hbar^2}{2m}(6\pi^2)^{2/3}$. To do so, write $N_\uparrow$ and $N_\downarrow$ in terms of a sum of the corresponding occupation numbers, and calculate the sums in the continuum limit, using the density of single-particle states for a single-component Fermi gas, $\mathcal{D}(\epsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\epsilon}$.

(c) (10 points) Calculate the kinetic energies per particle $U_{{\rm kin},\uparrow}/N_\uparrow$ and $U_{{\rm kin},\downarrow}/N_\downarrow$ of the two Fermi gases in terms of their respective Fermi energies.

We next assume that the spin up and down electrons interact such that the effective total interaction energy is

$$U_{\rm int} = g n_\uparrow n_\downarrow V \ ,$$

where $g > 0$ is an interaction strength. We will study the stability of the paramagnetic (spin-balanced) state $n_\uparrow = n_\downarrow$ by considering small deviations from it, i.e., $n_\uparrow = \frac{n}{2}(1+\delta)$ and $n_\downarrow = \frac{n}{2}(1-\delta)$, where δ is a small parameter.

(d) (10 points) Write down the total kinetic energy per unit volume, expressed only in terms of the densities $n_\uparrow$ and $n_\downarrow$ (and not the Fermi energies). Taylor-expand this expression to lowest nontrivial order in δ.

(e) (10 points) Write down the interaction energy per unit volume and Taylor-expand it to lowest nontrivial order in δ.

(f) (15 points) Use the results in parts (d) and (e) to write the total energy per unit volume of the system $u_{\rm tot}$ in the form $u_{\rm tot}(\delta) = c_0 + c_2\delta^2 + O(\delta^4)$. Provide the expressions for c_0 and c_2. Justify why all odd coefficients in the expansion are zero.

(g) (10 points) Find the critical value $g_{\rm c}$ of the interaction strength for which $c_2 = 0$.

(h) (25 points) Assuming that $c_4 > 0$, sketch $u_{\rm tot}$ versus δ for $g < g_{\rm c}$ and $g > g_{\rm c}$. Since the magnetization is $M \propto n_\uparrow - n_\downarrow \propto \delta$, use these sketches to determine the character of the equilibrium value of M [i.e., whether it is paramagnetic ($M = 0$) or ferromagnetic ($M \neq 0$)] as a function of g.

Hint: The equilibrium value of M is the one that minimizes $u_{\rm tot}$.

Useful formula: $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + ...$