QUALIFYING EXAMINATION, Part 1

9:00 – 11:30 am, Monday August 26, 2024

Attempt all parts of both problems.

Please begin your answer to each problem on a separate sheet, write your 3 digit code and the problem number on each sheet, and then number and staple together the sheets for each problem. Each problem is worth 100 points; partial credit will be given.

Calculators and cell phones may NOT be used.

Problem 1: Classical Mechanics I

Consider point masses $(m_1, m_2 \ldots m_N)$ located at positions $\vec{r}_i, \ldots \vec{r}_N$ with respect to an origin O.

(a) (10 points) Write an expression for $\vec{R}$, the center of mass (CM) coordinates.

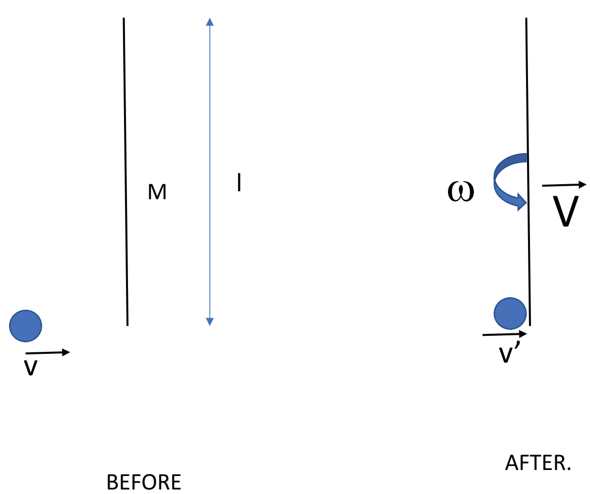
(b) (30 points) Writing $\vec{r}_i = \vec{R} + \vec{r'}_i$, where $\vec{r'}_i$ are the positions of the masses relative to the CM, show that the angular momentum $\vec{L}$ may be decomposed as

$$\vec{L} = \vec{L}_{CM} + \vec{L}_{\text{rel}} \ ,$$

where $\vec{L}_{CM} = M\vec{R} \times \dot{\vec{R}}$ is the angular momentum of the CM relative to the origin O ($M = \sum_i m_i$ is the total mass), and $\vec{L}_{\text{rel}}$ the angular momentum of the particles relative to the CM.

(c) (50 points) A rod of mass M and length l rests on a frictionless horizontal table (the (x, y) plane), with one end at $(0, 0)$ and the other end at $(0, l)$. It is struck at its lower end $(0, 0)$ by a point mass m moving at velocity $\vec{v} = v\hat{x}$ along the x-axis (see figure). Assuming the collision is *totally inelastic*, i.e., the mass sticks to the rod, what are the conservation equations? Using these and the equation in (b) for $\vec{L}$ determine $\vec{V} = V\hat{x}$ the velocity of the CM, ω, the angular velocity of the rod around the CM (positive if anti-clockwise), and $\vec{v'} = v'\hat{x}$ the linear velocity of the mass m right after the collision.

Hint: In addition to the conservation equations, use the equation relating v' to V and ω.



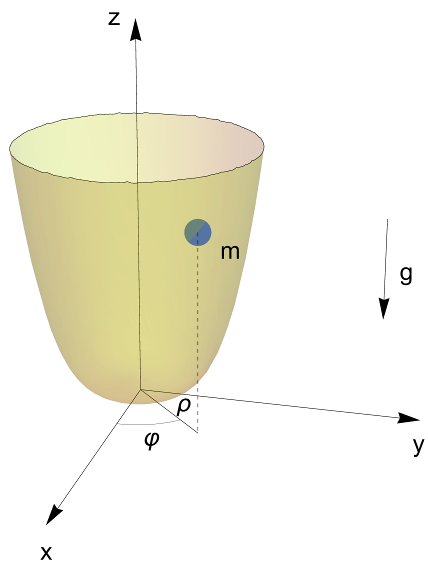
(d) (10 points) What changes qualitatively in the analysis in part (c) if the collision is elastic? Do not attempt any explicit calculations.

Problem 2: Classical Mechanics II

A bead (point particle) of mass m is confined by some unspecified force to move on the surface of a frictionless bowl. The bowl is in the shape of a surface of revolution about the z-axis which in cylindrical coordinates (ρ, φ, z) is given by the equation

$$z = \frac{\rho^4}{4a^3}\,,$$

where $a > 0$ has dimensions of length. In addition, the bead is in a constant gravitational field that points in the negative z direction, corresponding to a force $\vec{F} = -mg\hat{z}$, with $g > 0$ the gravitational acceleration.



(a) (20 points) Write the Lagrangian L for the bead, regarding the variables (ρ, φ) as the independent generalized coordinates.

(b) (20 points) Find the equations of motion for the generalized coordinates (ρ, φ).

(c) (20 points) Identify the continuous symmetries of the system, and find the corresponding first integrals (conserved quantities) of the motion implied by Noether's theorem. Give a brief physical interpretation of each constant of motion.

(d) (15 points) Suppose that the bead circles around the z-axis at some constant radius $\rho = \rho_c$. What is the relation between the period of the orbit and the radius ρ_c?

(e) (25 points) The bead originally in the circular orbit of part (d) is slightly displaced in the radial direction. Is the circular orbit stable against such a small perturbation? If so, find the frequency of small radial oscillations about $\rho = \rho_c$.

Hint: Eliminate $\dot{\varphi}$ in the equation of motion for ρ in terms of the angular momentum.

QUALIFYING EXAMINATION, Part 2

1:00 – 3:30 pm, Monday August 26, 2024

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Problem 1: Electromagnetism I

Consider a charged sphere of radius a and total charge Q. For each of the cases below, find the electric field inside and outside the sphere at a distance r from the center. Express your answer in terms of Q, a, and r.

Sketch the behavior of the electric field as a function of r for each case. In the sketches, use units of $\frac{Q}{4\pi\epsilon_0 a^2}$ for the electric field, and a for r.

(a) (20 points) A conducting sphere.

(b) (20 points) A uniformly charged sphere with a constant charge density ρ within its volume.

(c) (30 points) A sphere with a spherically symmetric charge density that varies radially as r^2.

(d) (30 points) A sphere with a spherically symmetric charge density that varies radially as $1/r^2$.

Hint: Recall that in SI units $\int \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$.

Problem 2: Electromagnetism II

A spatially uniform surface current, $\vec{K}$, flows in the x-y plane in the $\hat{x}$ direction:

$$\vec{K}(z=0,t) = K(t)\hat{x} \; .$$

The plane is electrically neutral (i.e., its net electric charge is zero everywhere).

(a) (20 points) First assume the current is constant in time, $K(t) = K_0$ for all times t. For this constant current, calculate the electric and magnetic field above the plane (i.e., for $z > 0$).

Next assume that at time $t < 0$ no current flows, but a constant current K_0 is turned on a time $t \geq 0$:

$$K(t) = \begin{cases} 0 & t < 0 \\ K_0 & t \geq 0 \end{cases}$$

(b) (30 points) Calculate the scalar potential, $\phi(\vec{x},t)$, and vector potential, $\vec{A}(\vec{x},t)$, as a function of position $\vec{x} = (x,y,z)$ in the region $z > 0$ at time $t \geq 0$.

Hint: Recall that in the Lorenz gauge, the potentials can be determined from the charge density, ρ, and current density, $\vec{J}$, in SI units as

$$\phi(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho(\vec{x}\,',t')}{|\vec{x}-\vec{x}\,'|} \, , \qquad \vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\vec{J}(\vec{x}\,',t')}{|\vec{x}-\vec{x}\,'|} \, ,$$

which depend on retarded time $t' = |\vec{x}-\vec{x}\,'|/c$, where c is the speed of light.

To calculate the vector potential, argue that it depends only on z, t and then calculate its value at $\vec{x} = (0,0,z)$ using polar coordinates r, φ for $\vec{x}\,'$ in the plane. The following indefinite integral may be useful:

$$\int \frac{u}{\sqrt{u^2 \pm a^2}} du = \sqrt{u^2 \pm a^2} + C \, .$$

(c) (20 points) Calculate the electric and magnetic field above the plane (i.e., for $z > 0$) as a function of time for the time-dependent current in part (b).

Recall that the electric and magnetic fields are given by $\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$ and $\vec{B} = \nabla \times \vec{A}$.

(d) (10 points) Find the power passing through a plane of area A parallel to the x-y plane at height $z > 0$ as a function of time for the constant current in part (a).

(e) (20 points) Find the power passing through a plane of area A parallel to the x-y plane at height $z > 0$ as a function of time for the time-dependent current in part (b).

Hint: Recall that in SI units the Poynting vector is given by

$$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}) \; .$$

QUALIFYING EXAMINATION, Part 3

9:00 – 11:30 am, Tuesday August 27, 2024

Attempt all parts of both problems.

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Problem 1: Quantum Mechanics I

An electron of mass m_e moves in a central electric potential $V(r)$ which is not purely Coulombic. (This could apply, for example, to a sodium atom for which the inner 10 electrons fill up all their allowed levels, leaving an outer, 11th electron moving in an effective potential $V(r)$ created by the nucleus and the inner 10 electrons.) It can be shown that the "spin-orbit" interaction between the magnetic moment of this outer electron and the magnetic field that it sees as it moves spatially, leads to an additive term in the Hamiltonian:

$$H_{LS} = \frac{1}{2m_e c^2} \frac{1}{r} \frac{dV(r)}{dr} \vec{L} \cdot \vec{S} \, ,$$

where $\vec{L}$ and $\vec{S}$ are, respectively, the orbital angular momentum and spin of the electron.

In the absence of this term, the energy eigenfunctions of the system may be written in the product form:

$$R_{nl}(r) \, Y_l^{m_l}(\theta, \varphi) \, \chi_\pm \, ,$$

where $\chi_\pm$ are two-component spinors describing the spin up/spin down electron, and n is a principal quantum number. These energy eigenfunctions are simultaneous eigenfunctions of $\vec{L}^2$, L_z, $\vec{S}^2$, and S_z.

(a) (20 points) In the absence of H_{LS}, the energy of the electron depends on n and l. What is the degeneracy of each level? Why is this degeneracy present? What are the possible values of the total angular momentum j of this system? Calculate the degeneracy of the level nl using the quantum numbers j and its projection m_j.

(b) (30 points) The energy shift due to H_{LS} may be expressed to first order in (degenerate) perturbation theory as the expectation value of H_{LS} with respect to normalized linear combinations of the above, degenerate eigenfunctions for which $\vec{L}^2$, $\vec{S}^2$, $\vec{J}^2$ and J_z are diagonal. (You do not need to write these linear combinations explicitly.) Is the operator $\vec{L} \cdot \vec{S}$ diagonal in this basis? Write an expression for the expectation value of $\vec{L} \cdot \vec{S}$ in the basis.

(c) (30 points) Using the result in (b), derive an expression for the energy shift of the energy eigenstates in terms of the total angular momentum of the states and the radial integral

$$\left\langle \frac{1}{r} \frac{dV(r)}{dr} \right\rangle_{nl} \equiv \int_0^\infty R_{nl}(r) \frac{1}{r} \frac{dV(r)}{dr} r^2 dr.$$

How many new energy levels appear? What is the degeneracy of each level? Why does this degeneracy remain?

(d) (20 points) For the sodium atom, suppose that the outer electron is in a d state ($l = 2$). Into how many levels is it split? What are the possible j values? Assuming that the above radial integral is positive, how are the corresponding energy levels ordered with respect to j?

Problem 2: Quantum Mechanics II

We set $\hbar = 1$ throughout the problem

Consider the hydrogen molecule ion H_2^+. Two protons of charge $+e$ are placed at the locations $(x, y, z) = (0, 0, \pm d)$. We will assume that the protons are fixed, so the Hamiltonian describes the motion of an electron shared between the two ions:

$$\hat{H} = -\frac{1}{2m_e}\nabla^2 - \frac{e^2}{\sqrt{\rho^2 + (z-d)^2}} - \frac{e^2}{\sqrt{\rho^2 + (z+d)^2}} \,,$$

where $\rho = \sqrt{x^2 + y^2}$. We will work with cylindrical coordinates ρ, φ, z, where $x = \rho\cos\varphi$ and $y = \rho\sin\varphi$.

In this problem we ignore the spin quantum number of the electron.

(a) (15 points) Argue that $[\hat{H}, \hat{L}_z] = 0$ but $[\hat{H}, \hat{L}_x] = [\hat{H}, \hat{L}_y] \neq 0$. Do not attempt to evaluate the commutators directly.

(b) (15 points) What is the symmetry group generated by $\hat{L}_z$? Does it imply any degeneracy in the spectrum of $\hat{H}$? Explain your answer.

(c) (20 points) Show that the eigenfunctions of $\hat{H}$ can be chosen to take the following form:

$$\psi_{\alpha m}(\rho, \varphi, z) = f_{\alpha m}(\rho, z)e^{im\varphi}$$

for integer m and some function $f_{\alpha m}$ that you do not need to find. The label α denotes any additional quantum numbers needed to specify the eigenstate (we will not need to know the details of α).

(d) (10 points) $\hat{H}$ is also invariant under the spatial reflection P_x, which acts on the Cartesian coordinates as

$$\hat{P}_x : \hat{x}, \hat{y}, \hat{z} \to -\hat{x}, \hat{y}, \hat{z} \,.$$

Show that $[\hat{H}, \hat{P}_x] = 0$.

(e) (10 points) Show that $\hat{P}_x\hat{L}_z = -\hat{L}_z\hat{P}_x$.

(f) (10 points) Show that $f_{\alpha m}(\rho, z)e^{im\varphi}$ and $f_{\alpha m}(\rho, z)e^{-im\varphi}$ are both eigenfunctions of $\hat{H}$ with the same energy.

From the above results, we conclude that that the energy levels $E_{\alpha m}$ satisfy $E_{\alpha m} = E_{\alpha,-m}$, so there is at least a 2-fold degeneracy for $m \neq 0$.

(g) (20 points) Now suppose that apart from the degeneracy given in part (f), there are no other degeneracies in the spectrum. Consider turning on a small electric field $-e\mathcal{E}\hat{x}$ along the x direction, which can be treated as a perturbation to $\hat{H}$. Use *degenerate perturbation theory* to calculate the splitting between $E_{\alpha,m=1}$ and $E_{\alpha,m=-1}$ to first-order in $\mathcal{E}$.

Useful formulas:

The Laplacian in cylindrical coordinates is

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}.$$

The z-component of the angular momentum operator is

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x.$$

QUALIFYING EXAMINATION, Part 4

1:00 – 3:30 pm, Tuesday August 27, 2024

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Problem 1: Statistical Mechanics I

Consider a box of area A with a lid fixed at height x, forming a three dimensional box, which contains N non-interacting indistinguishable particles each of mass m in equilibrium with a bath at inverse temperature $\beta = (k_B T)^{-1}$. Ignore the potential energy of the gas particles under gravity.

(a) (15 points) Calculate the *classical* canonical single-particle partition function, Z_1, as a function of height x and temperature T. Integrate explicitly over both coordinates and momenta.

(b) (15 points) Write the *classical* canonical partition function, Z_N for the N indistinguishable particles.

(c) (20 points) Next suppose that the lid of mass M is free to move vertically and is acted upon by a downward gravitational force $F = Mg$ so its potential energy at height x is given by $U = Fx$. Calculate the *classical* canonical partition function of the system of N particles plus the lid. Integrate explicitly over the position and momentum of the lid.

(d) (20 points) Express the average height $\langle x \rangle$ of the lid as an appropriate derivative of the logarithm of the partition function and use the partition function calculated in (c) to find $\langle x \rangle$.

(e) (15 points) Find the variance of the height of the lid, $\langle x^2 \rangle - \langle x \rangle^2$.

Hint: express the variance as a second logarithmic derivate of the partition function.

(f) (15 points) As temperature is lowered, the classical limit becomes less appropriate. At what temperature do you expect quantum effects to become important? Express this temperature in terms of the parameters F, m and A.

Hint: Quantum effects become important when the thermal deBroglie wavelength $\lambda = h/\sqrt{2\pi m k_B T}$ is of the order or larger than the average spacing between particles.

Useful formulas:

$$\int_{-\infty}^{\infty} du\, e^{-u^2} = \sqrt{\pi}\,, \qquad \int_{0}^{\infty} du\, u^N\, e^{-u} = N! \quad \text{for integer } N > 0\,.$$

Problem 2: Statistical Mechanics II

Consider a gas of noninteracting spin-1 bosons in a 3D cubic box of volume V. The gas is immersed in a spatially homogeneous magnetic field B along the $\hat{z}$ axis and is at equilibrium at temperature T. Including the linear Zeeman effect, the single-particle Hamiltonian is

$$H = \frac{\hbar^2 \mathbf{k}^2}{2m} - \mu_{\mathrm{b}} \sigma B,$$

where m is the boson mass and μ_{b} is its magnetic moment. The boson spin σ along direction $\hat{z}$ can take the values 0, ± 1. All calculations are to be performed in the grand-canonical ensemble.

(a) (15 points) Justify (no need to derive) that the occupation of the single-particle state of wavevector $\mathbf{k}$ and spin σ is

$$n_\sigma(\mathbf{k}) = \frac{1}{e^{\beta\left(\frac{\hbar^2 \mathbf{k}^2}{2m} - \mu_{\mathrm{b}}\sigma B - \mu\right)} - 1},$$

where μ is the chemical potential and $\beta = (k_{\mathrm{B}}T)^{-1}$.

(b) (20 points) Write down the expression for the total populations N_σ by summing over $\mathbf{k}$. Then write the continuum limit expressions by using the prescription $\sum_{\mathbf{k}} \to \frac{V}{(2\pi)^3} \int \mathrm{d}^3k$ and use the substitution $x^2 = \beta \frac{\hbar^2 k^2}{2m}$ to show that $N_\sigma / V = \lambda^{-3} F_{3/2}(\zeta e^{\beta \mu_{\mathrm{b}} \sigma B})$ where $\zeta = e^{\beta\mu}$ is the fugacity (see formulas below for the definition of λ and $F_{3/2}$). Finally, write explicitly the magnetization $M = \mu_{\mathrm{b}}(N_+ - N_-)$ for an arbitrary field B (where $N_\pm$ denote $N_{\pm 1}$).

(c) (20 points) Assume that B is small (compared to what?), expand the expression for M to first order in B. Use it to derive the zero-field magnetic susceptibility, $\chi = \frac{\partial M}{\partial B}|_{B=0}$.

(d) (20 points) Write the expression that determines the critical temperature for Bose-Einstein condensation T_{c} at $B = 0$ for a given total density of $\rho = (N_+ + N_- + N_0)/V$? How does χ behaves as T approaches T_{c} from above (at fixed density)?

(e) (15 points) What is μ in the BEC phase ($T < T_{\mathrm{c}}$) for small but non-zero B? What is the state that is macroscopically occupied?
Hint: The occupations $n_\sigma(\mathbf{k})$ must be positive for all σ.

(f) (10 points) Using (e), determine the macroscopic magnetization M in the BEC phase in the limit $B \to 0$. Are you surprised it is nonzero for $T < T_{\mathrm{c}}$?

Useful formulas:

λ is the thermal deBroglie wavelength defined by $\lambda = h/\sqrt{2\pi m k_{\mathrm{B}} T}$, where $h = 2\pi\hbar$.

The function $F_r(z)$ is defined by $F_r(z) = \frac{2}{\Gamma(r)} \int_0^\infty \frac{x^{2r-1}}{z^{-1}e^{x^2} - 1} \mathrm{d}x = \sum_{n=1}^\infty \frac{z^n}{n^r}$.

A useful limit: $\lim_{\zeta \to 1^-} F_{1/2}(\zeta) = \infty$.

For $\epsilon \ll 1$: $F_{3/2}[z(1+\epsilon)] \approx F_{3/2}(z) + F_{1/2}(z)\epsilon$.

Some useful values: $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(3/2) = \sqrt{\pi}/2$, $F_{3/2}(1) \approx 2.612$.