QUALIFYING EXAMINATION, Part 1

Solutions

Problem 1: Classical Mechanics I

(a) The kinetic energy T and potential energy V are given by

$$T = \frac{1}{2}M\,\dot{x}^2 + \frac{1}{2}\frac{M\,a^2}{2}(\dot{x}/a)^2 = \frac{3}{4}M\,\dot{x}^2$$

$$V = -Mgx\sin\alpha\ .$$

Thus the Lagrangian is

$$\mathcal{L} = T - V = \frac{3}{4}M\,\dot{x}^2 + Mgx\sin\alpha$$

$$\frac{3}{2}\ddot{x} - g\sin\alpha = 0$$

$$\ddot{x} = \frac{2}{3}g\sin\alpha$$

(b) The constraint of rolling without sliding is given by

$$\delta x = a\delta\theta\ .$$

Using the method of Lagrange multipliers, the variational principle reads $\delta\int_{t_1}^{t_2}[\mathcal{L} + \lambda(\delta x - a\delta\theta)]dt = 0$, where x and θ are considered as independent coordinates. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 + Mgx\sin\alpha\ .$$

The two equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) - \frac{\partial\mathcal{L}}{\partial x} = m\ddot{x} - Mg\sin\alpha = \lambda$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = I\ddot{\theta} = -\lambda a$$

(c) The constraint of rolling without sliding gives $\dot{x} = ad\dot{\theta}$ or

$$\ddot{x} = a\ddot{\theta}$$

Together with the equations in part (b), we find

$$M\ddot{x} - Mg\sin\alpha = -\frac{I}{a^2}\ddot{x}$$

or

$$\ddot{x} = \frac{Mg\sin\alpha}{M + I/a^2} = \frac{2}{3}g\sin\alpha\ .$$

(d) The work done in a displacement δx is

$$\delta W = \lambda\delta x = -f_s\delta x$$

where f_s is the static friction force. We find

$$f_s = -\lambda = \frac{I}{a^2}\ddot{x} = \frac{1}{3}Mg\sin\alpha\ .$$

f_s is pointing upward along the inclined plane.

Problem 2: Classical Mechanics II

(a)

$$\begin{aligned}
\gamma(v_1) &= \frac{1}{\sqrt{1-9/25}} = \frac{5}{4} \\
p_1 &= (\gamma(v_1)mc, \gamma(v_1)mv_1, 0, 0) = \left(\frac{5}{4}mc, \frac{3}{4}mc, 0, 0\right) \\
p_2 &= (mc, 0, 0, 0)
\end{aligned}$$

(b)

$$\begin{aligned}
p_f &= p_1 + p_2 = \left(\frac{9}{4}mc, \frac{3}{4}mc, 0, 0\right) \\
p_f \cdot p_f &= (M_f c)^2 = \left(\frac{9}{4}\right)^2 m^2c^2 - \left(\frac{3}{4}\right)^2 m^2c^2 = \frac{9}{2}m^2c^2 \\
M_f &= \frac{3}{\sqrt{2}}m \\
p_f^x &= \frac{3}{4}mc = \gamma(V_f)M_f V_f = \frac{3}{\sqrt{2}}m\frac{V_f}{\sqrt{1-V_f^2/c^2}} \\
V_f &= \frac{c}{3}
\end{aligned}$$

(c) In this reference frame we must have $|\vec{v}_1'| = |\vec{v}_2'| = v'$ by symmetry (or equivalently because their momentum must add to zero). The incoming particles have 4-momentum

$$
\begin{aligned}
p' &= (mc\gamma(v'), \pm mv'\gamma(v'), 0, 0) \\
&= \left(\frac{mc}{\sqrt{1 - v'^2/c^2}}, \pm \frac{mv'}{\sqrt{1 - v'^2/c^2}}, 0, 0 \right)
\end{aligned}
$$

Conservation of energy then gives

$$
\begin{aligned}
M_f c^2 &= \frac{2mc^2}{\sqrt{1 - v'^2/c^2}} \\
\frac{9}{2} &= \frac{4}{1 - v'^2/c^2} \\
v' &= \frac{c}{3}
\end{aligned}
$$

and $\vec{v}_1' = \frac{c}{3}\hat{x}, \vec{v}_2' = -\frac{c}{3}\hat{x}$.

(d) Using conservation of 4-momentum and $(p_{\gamma\pm})^2 = (p'_{\gamma\pm})^2 = 0$ in the S' frame, we have

$$
\begin{aligned}
p'_{\gamma+} + p'_{\gamma-} &= \frac{1}{c}((E'_{\gamma+} + E'_{\gamma-}), (E'_{\gamma+} - E'_{\gamma-}), 0, 0) = (M_f c, 0, 0, 0) \\
E'_{\gamma+} &= E'_{\gamma-} = \frac{1}{2} M_f c^2 \\
\frac{E'_{\gamma+}}{E'_{\gamma-}} &= 1
\end{aligned}
$$

while in the S frame we have

$$
\begin{aligned}
p_{\gamma+} + p_{\gamma-} &= \frac{1}{c}((E_{\gamma+} + E_{\gamma-}), (E_{\gamma+} - E_{\gamma-}), 0, 0) = \left(\frac{9}{4}mc, \frac{3}{4}mc, 0, 0 \right) \\
E_{\gamma+} = 2E_{\gamma-} &= \frac{3}{2}mc^2 \\
\frac{E_{\gamma+}}{E_{\gamma-}} &= 2
\end{aligned}
$$