QUALIFYING EXAMINATION, Part 2

Solutions

Problem 1: Electromagnetism I

(a) We choose polar coordinates centered at the spherical bubble. The interface between the two materials is then the surface $r = a$.

Integrating $\nabla \cdot \mathbf{J} = 0$ inside the region bounded by a small "Gaussian pillbox" that stradles the two media implies that the normal component of $\mathbf{J}$ is continuous across the surface $r = a$,

$$J_\perp(r = a^-) = J_\perp(r = a^+).$$

Integrating $\nabla \times \mathbf{E} = 0$ over a surface bounded by an infinitesimal closed loop that intersects $r = a$ yields the condition that the tangential electric field must be continuous $E_\parallel(r = a^-) = E_\parallel(r = a^+)$. By Ohm's law, $\mathbf{J} = \sigma \mathbf{E}$, we find the condition

$$J_\parallel(r = a^-)/\sigma_1 = J_\parallel(r = a^+)/\sigma_2. \tag{1}$$

on the tangential component of the current density.

(b) In terms of the electrostatic potential $\mathbf{E} = -\nabla \Phi$, eq. () implies the jump condition on the normal derivative of Φ at the spherical surface $r = a$,

$$\sigma_1 \hat{r} \cdot \nabla \Phi(r = a^-) = \sigma_2 \hat{r} \cdot \nabla \Phi(r = a^+). \tag{2}$$

On the other hand, $\nabla \times \mathbf{E} = 0$ implies that the tangential derivative of Φ must be continuous. Given that the gradient of Φ has at worse a step function discontinuity at $r = a$ then implies that Φ must be continuous across $r = a$:

$$\Phi(r = a^-) = \Phi(r = a^+). \tag{3}$$

(c) The steady current coming in from infinity is equivalent to, by Ohm's law, an applied electric field $\mathbf{E}_0 = J_0 \hat{z}/\sigma_2$, corresponding to an electrostatic potential $\Phi_0(r, \theta) = -(J_0/\sigma_2) r \cos\theta$. Given the azimuthal symmetry of the problem, the general solution in the exterior consistent with $\mathbf{E}(r \to \infty) = \mathbf{E}_0$ takes the form

$$\Phi(r > a, \theta) \equiv \Phi_>(r, \theta) = -(J_0/\sigma_2) r P_1(\cos\theta) + \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta),$$

where the first term is the external potential $\Phi_0(r, \theta)$ and the second characterizes the response of the medium. In the interior, the general solution which is regular at $r = 0$ is

$$\Phi(r \le a, \theta) \equiv \Phi_<(r, \theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta).$$

The unknown coefficients A_ℓ, B_ℓ can be found by applying the boundary conditions in part (b). Note that because the external potential has only an $\ell = 1$ (dipole) component, rotational invariance immediately implies that $A_{\ell\neq 1} = B_{\ell\neq 1} = 0$, which can also be verified by direct calculation. For the $\ell = 1$ terms, eq. (3) implies that

$$aA_1 = -\frac{J_0 a}{\sigma_2} + \frac{B_1}{a^2},$$

while eq. (2) yields

$$\sigma_1 A_1 = -J_0 - \frac{2\sigma_2 B_1}{a^3},$$

with solution $A_1 = -3J_0/(\sigma_1 + 2\sigma_2)$ and $B_1 = \left(\frac{\sigma_1-\sigma_2}{\sigma_1+2\sigma_2}\right) J_0 a^3/\sigma_2$.

(d) Using the potential Φ found in (c), we can calculate the electric field from $\mathbf{E} = -\nabla\Phi$. By Gauss' law, the induced surface charge density $\Sigma(\theta)$ at $r = a$ is proportional to the discontinuity in the component of $\mathbf{E}$ normal to the interface:

$$\kappa\Sigma(\theta) = E_\perp(r = a^+) - E_\perp(r = a^-) \ ,$$

where $\kappa_{cgs} = 4\pi$,$\kappa_{LH} = 1$,$\kappa_{SI} = 1/\epsilon_0$.

(e) The power per unit volume dissipated by an Ohmic conductor is given by $\mathbf{J}\cdot\mathbf{E} = \sigma\mathbf{E}^2$. The total ohmic heat produced inside the sphere by the current flow

$$P = \int_{r\leq a} d^3\mathbf{x}\sigma_1\mathbf{E}^2 \ ,$$

where $\mathbf{E}$ is the field inside the sphere.

Problem 2: Electromagnetism II

(a) In the quasistatic limit, we need to ensure that the charges/currents change slowly compared to the light transit time across the geometry, i.e.,

$$\omega \ll \frac{c}{l}$$

or

$$\frac{\omega l}{c} \ll 1$$

(b) The voltage sets up a current, $I(t) = \frac{V_0}{R_0}\cos(\omega t)$ flowing to the right along the inner cylinder and returning along the outer cylinder. Note, in order to keep the form of the current simple, we assume the cable inductance and frequency are small, i.e. $\omega L \ll R_0$, so that their effect on the current can be neglected. From Ampere's law this gives a magnetic field

$$\mathbf{B}(t) = \begin{cases} \frac{\mu_0}{2\pi}\frac{V_0}{R_0}\cos(\omega t)\frac{\hat{\phi}}{r} & a < r < b \\ 0 & otherwise \end{cases}$$

The changing E-field can also in principle induce a magnetic field. However, in the quasistatic limit we can neglect these contributions to B since those at linear order in ω contain the small parameter $\omega l/c$.

(c) In addition to the current, the voltage source sets up a net charge per unit length, $Q/l = (C/l)V_0$ for capacitance per unit length C/l (with opposite charge on the outer conductor). From Gauss's Law, between the conductors

$$\mathbf{E} = \frac{Q/l}{2\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r}$$

To find the capacitance, we can integrate the field: $V = -\int_b^a \mathbf{E}\cdot d\mathbf{l}$, which gives

$$V = \frac{Q/l}{2\pi\epsilon_0}\ln(b/a)$$

so substituting in above, we find

$$\mathbf{E}(t) = \begin{cases} \frac{V_0}{\ln(b/a)}\cos(\omega t)\frac{\hat{\mathbf{r}}}{r} & a < r < b \\ 0 & otherwise \end{cases}$$

There is also an induced term linear in ω from the changing B-field. We apply Faraday's law to a loop with edges parallel to the cylinder axis. For such a loop with length L and one edge at distance outside the outer cylinder where the fields must be zero, and the other at $a < r < b$, the integral form of Faraday's law gives the magnitude of the induced field (to linear order in ω):

$$EL = \frac{\mu_0 V_0 \omega \sin(\omega t) L}{2\pi R_0} \ln(b/r)$$

This field points in the positive $\hat{\mathbf{z}}$ direction in order to oppose the change in current. Similarly, for a loop extending inside the inner cylinder the flux through this part of the loop is zero, so the E-field must be constant inside $r < a$ and equal to that on the surface of the inner cylinder.

This implies an additional contribution to the E-field which is, e.g., $\frac{\mu_0}{2\pi}\frac{V_0}{R_0}\omega \sin(\omega t) \ln(\frac{b}{r})\hat{\mathbf{z}}$ in the region $a < r < b$. However, this contribution does not satisfy the boundary condition that the parallel electric field $\mathbf{E}^{||}$ at the inner conductor surface ($r = a$) should vanish. This can be corrected by allowing for a z dependence of the charge distribution along the conductor such that the net field $\mathbf{E}^{||} = 0$ at $r = a$. An example is the TEM mode in a coaxial transmission line in which the electric field is purely radial.

(d) The energy flux through a surface is given by the integral of the Poynting vector, $\int \mathbf{S} \cdot d\mathbf{a}$, where

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) \, .$$

Using the zeroth order term in $\mathbf{E}$ found in part (c), we have

$$\mathbf{S}(t) = \begin{cases} \frac{V_0^2}{2\pi R_0}\left(\frac{\cos^2(\omega t)}{\ln(b/a)}\frac{\hat{\mathbf{z}}}{r^2}\right) & a < r < b \\ 0 & otherwise \end{cases}$$

Through a cross section perpendicular to the cable axis, we have

$$\begin{aligned} \int \mathbf{S} \cdot d\mathbf{a} &= \frac{V_0^2}{2\pi R_0}\left(\frac{\cos^2(\omega t)}{\ln(b/a)}\right)\int_a^b \frac{1}{r^2} r dr d\phi \\ &= \frac{V_0^2}{R_0}\cos^2(\omega t) \, . \end{aligned}$$

This result is independent of z position and also gives the power delivered to the resistor.