QUALIFYING EXAMINATION, Part 4

Solutions

Problem 1: Quantum Mechanics I

(a) ϕ_1 has even parity, ϕ_2 has odd parity.

(b,c) The normalized wave functions are

$$\psi(x_1, x_2) = \frac{1}{\sqrt{2}} \left[\phi_1(x_1)\phi_2(x_2) \pm \phi_2(x_1)\phi_1(x_2)\right],$$

where $+/-$ for boson/fermion.

(d,e) The probability is given by

$$\begin{aligned} P_{LL} &= \int_{-L/2}^{0} dx_1 \int_{-L/2}^{0} dx_2 \, |\psi(x_1, x_2)|^2 \\ &= \frac{1}{2} \left[2 \int_{-L/2}^{0} dx_1 \, \phi_1^2(x_1) \int_{-L/2}^{0} dx_2 \, \phi_2^2(x_2) \pm 2 \left(\int_{-L/2}^{0} dx \, \phi_1(x)\phi_2(x) \right)^2 \right] \end{aligned}$$

Since both $\phi_1^2(x)$ and $\phi_2^2(x)$ are even functions of x, we have

$$\int_{-L/2}^{0} dx \, \phi_a^2(x) = \frac{1}{2} \int_{-L/2}^{L/2} dx \, \phi_a^2(x) = \frac{1}{2}, \; a = 1, 2.$$

The other integral is found to be

$$\int_{-L/2}^{0} dx \, \phi_1(x)\phi_2(x) = \int_{0}^{L/2} dx \, \phi_1(-x)\phi_2(-x) = -\int_{0}^{L/2} dx \, \phi_1(x)\phi_2(x) = -\frac{4}{3\pi}.$$

Therefore we have

$$P_{LL} = \frac{1}{4} \pm \left(\frac{4}{3\pi} \right)^2.$$

(f) The probability of finding the particles in opposite halves is just $1 - P_{LL} - P_{RR}$. By symmetry, it is clear that $P_{RR} = P_{LL}$, so

$$P_{LR} + P_{RL} = 1 - 2P_{LL} = \frac{1}{2} \mp 2 \left(\frac{4}{3\pi} \right)^2.$$

Problem 2: Quantum Mechanics II

(a) Straightforward algebra gives

$$\hat{H} = \frac{\mathbf{P}^2}{4m} + \frac{\mathbf{p}^2}{m} + V(\mathbf{r}).$$

Since $\mathbf{R}$ does not appear in H (due to translation invariance) and $[\mathbf{P}, H] = 0$, the center of mass DOF decouples completely.

(b) The eigenfunctions $\psi_l(\phi)$ of l satisfies

$$-i\frac{d}{d\phi}\psi_l(\phi) = l\psi_l(\phi),$$

the solution of which is $\frac{1}{\sqrt{2\pi}}e^{il\phi}$. To satisfy the boundary condition, we must have

$$e^{il(\phi+2\pi)} = -e^{il\phi}.$$

In other words $e^{2\pi il} = -1$, so l take values $\pm 1/2, \pm 3/2, \cdots$, i.e. half-integers.

(c) Because only l^2 appears in H_l, $H_l = H_{-l}$. So $E_{ln} = E_{-l,n}$ and $f_{ln} = f_{-l,n}$. The only l for which $l = -l$ is $l = 0$, but this is not allowed by the quantization condition in (b). So each level is two-fold degenerate, with the two eigenfunctions given by $\frac{1}{\sqrt{2\pi}} f_{ln}(r)e^{\pm il\phi}$. The eigenstates will be denoted by $|l, n\rangle$ below.

(d) Since $\hat{\Theta}$ is complex conjugation in the coordinate basis, the eigenfunction $f_{ln}(r)e^{il\phi}$ becomes $f_{ln}(r)e^{-il\phi}$ (note that $f_{ln}(r)$ is real). So

$$\hat{\Theta}\,|l, n\rangle = |-l, n\rangle\,.$$

$\hat{\Theta}^2 = 1$ is evident because $\psi^{**} = \psi$.

(e) First we find how $\hat{C}$ acts on eigenstates. Since $\hat{C}$ takes ϕ to $\phi + \pi$, we have $e^{il\phi} \to e^{il(\phi+\pi)} = e^{il\pi}e^{il\phi}$. Combine with the action of $\hat{\Theta}$, we have

$$\tilde{\hat{\Theta}}\,|l, n\rangle = \hat{\Theta}e^{il\pi}\,|l, n\rangle = e^{-il\pi}\,|-l, n\rangle\,.$$

(f)

$$\tilde{\Theta}^2\,|l, n\rangle = \tilde{\Theta}e^{-il\pi}\,|-l, n\rangle = e^{il\pi}\tilde{\Theta}\,|-l, n\rangle = e^{2il\pi}\,|l, n\rangle\,.$$

Because l takes half-integer values, $e^{2\pi il} = -1$. So $\tilde{\Theta}^2 = -1$. $|l, n\rangle$ and $|-l, n\rangle$ form a Kramers doublet.

(g) We treat $\epsilon V_0(r)\cos^2\phi$ as a perturbation, which can be written as

$$\epsilon V_0(r)\cos^2\phi = \frac{1}{2}\epsilon V_0(r) + \frac{1}{4}\epsilon V_0(r)(e^{2i\phi} + e^{-2i\phi}).$$

At first order, the perturbation only changes l by 0 (the first term) or ± 2 (the second term). Because the two degenerate eigenstates $|\pm l, n\rangle$ differ by odd $\Delta l = 1, 3, 5, \ldots$, they are not mixed by the perturbation. So both eigenstates $|\pm l, n\rangle$ have a first-order shift given by

$$\Delta E_{ln} = \frac{1}{2}\epsilon \langle l, n|V_0|l, n\rangle = \frac{1}{2}\epsilon \int_0^\infty rdr\, V_0(r) f_{ln}^2(r).$$

Obviously $\Delta E_{ln} = \Delta E_{-l,n}$.

The perturbed potential is real and invariant under $\phi \to \phi + \pi$, so it preserves the $\hat{\tilde{\Theta}}$ symmetry. The degeneracy is therefore guaranteed by Kramers' theorem.