Problem 1: Quantum Mechanics I

(a) \( \phi_1 \) has even parity, \( \phi_2 \) has odd parity.

(b,c) The normalized wave functions are

\[
\psi(x_1, x_2) = \frac{1}{\sqrt{2}} \left[ \phi_1(x_1) \phi_2(x_2) \pm \phi_2(x_1) \phi_1(x_2) \right],
\]

where +/− for boson/fermion.

(d,e) The probability is given by

\[
P_{LL} = \int_{-L/2}^{L/2} dx_1 \int_{-L/2}^{L/2} dx_2 |\psi(x_1, x_2)|^2 = 2 \int_{-L/2}^{0} dx_1 \phi_1^2(x_1) \int_{-L/2}^{0} dx_2 \phi_2^2(x_2) \pm 2 \left( \int_{-L/2}^{0} dx \phi_1(x) \phi_2(x) \right)^2.
\]

Since both \( \phi_1^2(x) \) and \( \phi_2^2(x) \) are even functions of \( x \), we have

\[
\int_{-L/2}^{0} dx \phi_a^2(x) = \frac{1}{2} \int_{-L/2}^{L/2} dx \phi_a^2(x) = \frac{1}{2}, \quad a = 1, 2.
\]

The other integral is found to be

\[
\int_{-L/2}^{0} dx \phi_1(x) \phi_2(x) = \int_{0}^{L/2} dx \phi_1(-x) \phi_2(-x) = -\int_{0}^{L/2} dx \phi_1(x) \phi_2(x) = \frac{4}{3\pi}.
\]

Therefore we have

\[
P_{LL} = \frac{1}{4} \pm \left( \frac{4}{3\pi} \right)^2.
\]

(f) The probability of finding the particles in opposite halves is just \( 1 - P_{LL} - P_{RR} \). By symmetry, it is clear that \( P_{RR} = P_{LL} \), so

\[
P_{LR} + P_{RL} = 1 - 2P_{LL} = \frac{1}{2} \pm 2 \left( \frac{4}{3\pi} \right)^2.
\]
Problem 2: Quantum Mechanics II

(a) Straightforward algebra gives
\[ \hat{H} = \frac{\hat{P}^2}{4m} + \frac{\hat{P}^2}{m} + V(\mathbf{r}). \]
Since \( \mathbf{R} \) does not appear in \( H \) (due to translation invariance) and \([\hat{P}, H] = 0\), the center of mass DOF decouples completely.

(b) The eigenfunctions \( \psi_l(\phi) \) of \( l \) satisfies
\[ -i \frac{d}{d\phi} \psi_l(\phi) = l \psi_l(\phi), \]
the solution of which is \( \frac{1}{\sqrt{2\pi}} e^{il\phi} \). To satisfy the boundary condition, we must have
\[ e^{il(\phi+2\pi)} = -e^{il\phi}. \]
In other words \( e^{2\pi il} = -1 \), so \( l \) take values \( \pm 1/2, \pm 3/2, \cdots \), i.e. half-integers.

(c) Because only \( l^2 \) appears in \( H_l \), \( H_l = H_{-l} \). So \( E_{ln} = E_{-l,n} \) and \( f_{ln} = f_{-l,n} \). The only \( l \) for which \( l = -l \) is \( l = 0 \), but this is not allowed by the quantization condition in (b). So each level is two-fold degenerate, with the two eigenfunctions given by \( \frac{1}{\sqrt{2\pi}} f_{ln}(r)e^{\pm il\phi} \). The eigenstates will be denoted by \( |l, n\rangle \) below.

(d) Since \( \hat{\Theta} \) is complex conjugation in the coordinate basis, the eigenfunction \( f_{ln}(r)e^{il\phi} \) becomes \( f_{ln}(r)e^{-il\phi} \) (note that \( f_{ln}(r) \) is real). So
\[ \hat{\Theta} |l, n\rangle = |l, n\rangle. \]
\( \hat{\Theta}^2 = 1 \) is evident because \( \psi^{**} = \psi \).

(e) First we find how \( \hat{C} \) acts on eigenstates. Since \( \hat{C} \) takes \( \phi \) to \( \phi + \pi \), we have \( e^{il\phi} \to e^{i(l+\pi)\phi} = e^{i\pi} e^{il\phi} \). Combine with the action of \( \hat{\Theta} \), we have
\[ \hat{\Theta} |l, n\rangle = \hat{\Theta} e^{i\pi} |l, n\rangle = e^{-il\pi} |l, n\rangle. \]

(f)
\[ \hat{\Theta}^2 |l, n\rangle = \hat{\Theta} e^{-il\pi} |l, n\rangle = e^{il\pi} \hat{\Theta} |l, n\rangle = e^{2il\pi} |l, n\rangle. \]
Because \( l \) takes half-integer values, \( e^{2\pi il} = -1 \). So \( \hat{\Theta}^2 = -1 \). \( |l, n\rangle \) and \( |l, n\rangle \) form a Kramers doublet.
(g) We treat $\epsilon V_0(r) \cos^2 \phi$ as a perturbation, which can be written as

$$\epsilon V_0(r) \cos^2 \phi = \frac{1}{2} \epsilon V_0(r) + \frac{1}{4} \epsilon V_0(r) (e^{2i\phi} + e^{-2i\phi}).$$

At first order, the perturbation only changes $l$ by 0 (the first term) or $\pm 2$ (the second term). Because the two degenerate eigenstates $|\pm l, n\rangle$ differ by odd $\Delta l = 1, 3, 5, \ldots$, they are not mixed by the perturbation. So both eigenstates $|\pm l, n\rangle$ have a first-order shift given by

$$\Delta E_{ln} = \frac{1}{2} \epsilon \langle l, n|V_0|l, n\rangle = \frac{1}{2} \epsilon \int_0^\infty rdr V_0(r) f_{ln}^2(r).$$

Obviously $\Delta E_{ln} = \Delta E_{-l,n}$.

The perturbed potential is real and invariant under $\phi \rightarrow \phi + \pi$, so it preserves the $\hat{\Theta}$ symmetry. The degeneracy is therefore guaranteed by Kramers’ theorem.