QUALIFYING EXAMINATION, Part 1

Solutions

Problem 1: Classical Mechanics I

(a) The center of force can be taken at the center of the planet. Calculating the angular momentum relative to the center of the planet for the distant asteroid:

$$l = |\mathbf{r} \times \mathbf{p}| = rmv \sin\varphi = bmv$$

where we have used that the angle between the radial vector and asteroid momentum is given by $\sin\varphi = b/r$.

(b) Since $m \ll M$, the reduced mass is

$$\mu = \frac{mM}{m+M} \approx m \; .$$

The effective potential is then

$$V_{eff}(r) = -\frac{GmM}{r} + \frac{(bmv)^2}{2mr^2} = m\left[\frac{(bv)^2}{2r^2} - \frac{GM}{r}\right] \; .$$

(c) The asteroid will just hit the planet if the distance of closest approach is $r_m = R$. Using energy conservation, we can equate the energy when far from the planet to the energy at the point of closest approach to solve for the impact parameter. Far away (for $r \to \infty$), the energy is $E = mv^2/2$. For a closest approach equal to R (and taking $\dot{r}$=0 as in the hint), we have $E = V_{eff}(R)$. Equating these, we find

$$\frac{1}{2}mv^2 = m\left[\frac{(bv)^2}{2R^2} - \frac{GM}{R}\right]$$
$$\Rightarrow b^2 = R^2 + \frac{2GMR}{v^2} \; .$$

Thus the asteroid will hit the surface for any impact parameter such that

$$b \leq R\sqrt{1 + \frac{2GM}{Rv^2}} \; .$$

As expected, this approaches R for high initial velocity, while for $v^2 < 2GM/R$ the asteroid will collide with the planet for a larger range of impact parameters.

(d) To find the scattering angle, we use the formula given with $V(u) = -GMmu$ and $E = mv^2/2$. Making the variable transformation $x = bu$, the integral can be rewritten as in the hint with $\alpha = -2GM/(bv^2)$. The scattering angle is then

$$\theta = 2\arctan\left(\frac{GM}{bv^2}\right) \; .$$

(e) Substituting the result in part (c) for the impact parameter where the minimum approach is R, we find the scattering angle when the asteroid just misses the planet

$$\theta = 2\arctan\left(\frac{GM}{v^2R\sqrt{1+\frac{2GM}{v^2R}}}\right) .$$

We next check that the limits make sense:

- When $v \to \infty$, then $\theta \to 2\arctan(0) = 0$, i.e., for high velocity the asteroid keeps going with no deflection even at b for which the asteroid just barely misses the planet.
- When $v \to 0$ but for sufficiently large b such that the asteroid just barely misses the planet, then $\theta \to 2\arctan(\infty) = 2(\pi/2) = \pi$, i.e., the asteroid is completely backscattered. In this case it will circle the planet at radius just above its surface and return in the direction it came from.

Problem 2: Classical Mechanics II

(a) The moment of inertia of the compound pendulum can be calculated in polar coordinates ρ, φ

$$I = \frac{2m}{\alpha R^2}\int_0^R \rho\,\mathrm{d}\rho \int_{-\alpha/2}^{\alpha/2} \mathrm{d}\varphi \rho^2 = \frac{mR^2}{2} ,$$

where $\frac{2m}{\alpha R^2}$ is the areal density. Note that the area of the system is given by $\alpha/(2\pi)$ times the area of the disk πR^2, i.e., area $= \alpha R^2/2$.

The moment of inertia can also be calculated as $\alpha/(2\pi)$ times the moment of inertia of the disk.

(b) The kinetic energy is

$$T = \frac{1}{2}I\dot{\theta}^2 = \frac{1}{4}mR^2\dot{\theta}^2 .$$

(c) To find the potential energy, we can consider the mass of the system to be concentrated at its center of mass. The distance of the center of mass from the point of support can be calculated in polar coordinates ρ, φ

$$r_{\mathrm{COM}} = \frac{2m}{\alpha R^2}\int_0^R \rho\,\mathrm{d}\rho \int_{-\alpha/2}^{\alpha/2} \mathrm{d}\varphi \rho \cos\varphi = \frac{2R}{3}\frac{\sin(\alpha/2)}{\alpha/2} .$$

The potential energy is then

$$V = mgr_{\mathrm{COM}}(1-\cos\theta) = \frac{2mgR}{3}\frac{\sin(\alpha/2)}{\alpha/2}(1-\cos\theta) .$$

(d) Using the Lagrangian formulation

$$\mathcal{L} = T - V = \frac{1}{2}\frac{mR^2}{2}\dot{\theta}^2 - \frac{2mgR}{3}\frac{\sin(\alpha/2)}{\alpha/2}(1-\cos\theta),$$

we derive below the Euler-Lagrange equations for θ.

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{mR^2}{2}\dot{\theta} \; .$$

$$\frac{\mathrm{d}}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{mR^2}{2}\ddot{\theta}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -\frac{2mgR}{3}\frac{\sin(\alpha/2)}{\alpha/2}\sin\theta$$

$$\frac{\mathrm{d}}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = \frac{mR^2}{2}\ddot{\theta} + \frac{2mgR}{3}\frac{\sin(\alpha/2)}{\alpha/2}\sin\theta = 0,$$

or

$$\ddot{\theta} + \frac{4g}{3R}\frac{\sin(\alpha/2)}{\alpha/2}\sin\theta = 0 \; .$$

In the limit of small oscillations $\sin\theta \approx \theta$ and

$$\omega_0 = \sqrt{\frac{4g}{3R}\frac{\sin(\alpha/2)}{\alpha/2}} \; .$$

(e) A simple pendulum has a frequency $\omega_0 = \sqrt{\frac{g}{R}}$. For $\alpha \to 0$, the compound pendulum swings faster. For $\alpha \to 2\pi$ (a complete disk suspended at its center), $\omega_0 = 0$. The frequencies coincide when

$$\frac{\sin(\alpha/2)}{\alpha/2} = \frac{3}{4} \; .$$

(f) Using conservation of energy, or multiplying the equation of motion by $\dot{\theta}$ and integrating with respect to time gives

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega_0\sqrt{2}\sqrt{\cos\theta - \cos\theta_0} \; ,$$

or

$$\mathrm{d}t = \frac{\mathrm{d}\theta}{\omega_0\sqrt{2}\sqrt{\cos\theta - \cos\theta_0}}.$$

Integrating between 0 and θ_0 corresponds to one quarter of the full period T, i.e.,

$$T = 4\int_0^{\theta_0} \frac{\mathrm{d}\theta}{\omega_0\sqrt{2}\sqrt{\cos\theta - \cos\theta_0}}.$$

As $\theta_0 \to \pi$ the pendulum will approach an unstable equilibrium configuration and we expect the period $T \to \infty$. This can be confirmed by explicitly doing the integral.