QUALIFYING EXAMINATION, Part 2

Solutions

Problem 1: Electromagnetism I

(a) We can straightforwardly compute the potential of the configuration by superimposing the potentials of two point charges

$$V(r) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right),$$

where

$$r_\pm = r\sqrt{1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}}\,.$$

Taylor expanding the result, we find that the $1/r$ contribution to the potential cancels, and keeping only the leading term, we have

$$V(r) = \frac{qd}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}\,.$$

This potential falls off as $1/r^2$.

(b) Defining the $\hat{y}$ direction to be along the direction of the dipole, we have

$$\vec{p} = q\frac{d}{2}\,\hat{y} + (-q)\frac{d}{2}(-\hat{y}) = qd\,\hat{y}\,.$$

Substituting this result into the formula for the potential of a dipole, we have

$$V_{\rm dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{qd}{4\pi\epsilon_0}\frac{\hat{y}\cdot\hat{r}}{r^2} = \frac{qd}{4\pi\epsilon_0}\frac{\cos\theta}{r^2},$$

which agrees with the result from the explicit calculation in (a).

(c) The dipole moment for both configurations vanishes.

(d) The quadrupole moment tensor takes the general form

$$Q = \begin{pmatrix} Q_{xx} & Q_{xy} & Q_{xz} \\ Q_{yx} & Q_{yy} & Q_{yz} \\ Q_{zx} & Q_{zy} & Q_{zz} \end{pmatrix},$$

and is a symmetric, traceless tensor.

Here we take the $\hat{z}$ vector to point out of the plane. We first note that from the structure of the configuration, we have

$$Q_{iz} = Q_{zi} = 0\,.$$

Second, we note that

$$Q_{xx} = Q_{yy} = 0\,.$$

Therefore

$$Q = \begin{pmatrix} 0 & Q_{xy} & 0 \\ Q_{xy} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,.$$

We then have

$$V_{\text{quad}} = \frac{1}{4\pi\epsilon_0}\frac{1}{2r^3}\begin{pmatrix}\sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta\end{pmatrix}\begin{pmatrix} 0 & Q_{xy} & 0 \\ Q_{xy} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix}\sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta\end{pmatrix}\,.$$

Simplifying the result, we find

$$V_{\text{quad}} = \frac{1}{4\pi\epsilon_0}\frac{1}{2r^3}Q_{xy}\sin^2\theta\sin(2\varphi)\,.$$

The explicit value of Q_{xy} does not need to be computed, but is non-vanishing.

(e) The quadrupole moment for this configuration vanishes by symmetry. To see this, we note that in the definition of Q_{ij}, the δ_{ij} term vanishes since the sum of the charges is zero, and they are all equidistant from the origin. The $r_i r_j$ term vanishes since for every charge, there is an opposite charge in the mirrored position $r_i \to -r_i$, $r_j \to -r_j$. The mirroring does not change the sign of the bilinear $r_i r_j$, so the mirrored charges all cancel in the sum defining the quadrupole moment.

However, this configuration has a non-trivial octupole moment. We therefore expect that the potential will fall off as $1/r^4$.

Problem 2: Electromagnetism II

(a) At $t \geq T$, the external magnetic field is zero, so the only contribution to the magnetic field is from the rotating cylinder.

The magnetic field outside the cylinder is 0. The field $\vec{B}$ inside the cylinder is parallel to the symmetry axis of the cylinder. Using Ampere's law for a rectangular loop with sides of length l parallel to the symmetry axis of the cylinder such that one side is inside the cylinder and another outside, we have

$$Bl = \mu_0 I_{enc} = \mu_0(\sigma v)l = \mu_0(\frac{\lambda}{2\pi R}\omega R)l \ ,$$

or

$$\vec{B} = \frac{\mu_0 \lambda \omega}{2\pi}\hat{z}$$

(b) The initial angular momentum density store in the fields is given by $\vec{l}_{EM} = \epsilon_0 \vec{r} \times (\vec{E} \times \vec{B})$.

We can use Gauss's law to solve for the electric field $\vec{E}$ by considering a cylindrical surface of radius ρ. The charges cancel outside of the cylindrical shell, so the only region that contributes is inside the shell, $\rho < R$. We find

$$E \ 2\pi\rho l = \frac{1}{\epsilon_0}(-\lambda l) \ ,$$

or

$$\vec{E} = -\frac{\lambda}{2\pi\rho\epsilon_0}\hat{\rho} \ .$$

Initially the magnetic field is $\vec{B} = B_{ex}\hat{z}$. Thus the initial angular momentum density in the fields is

$$\vec{l}_{EM} = \epsilon_0 \vec{r} \times \left(\frac{-\lambda}{2\pi\rho\epsilon_0}\hat{\rho} \times B_{ex}\hat{z}\right) \ .$$

Using the formula for the vector triple product, we find

$$\vec{l}_{EM} = \frac{-\lambda}{2\pi\rho}(\vec{r} \cdot B_{ex}\hat{z})\,\hat{\rho} - B_{ex}\left(\vec{r} \cdot \frac{-\lambda}{2\pi\rho}\hat{\rho}\right)\,\hat{z} = -\frac{\lambda}{2\pi\rho}B_{ex} z\,\hat{\rho} + \frac{\lambda}{2\pi}B_{ex}\,\hat{z} \ .$$

When integrating to find the total angular momentum, the first term on the r.h.s. will vanish and only the second term will contribute (in any case, we are only interested in the angular momentum along the z axis). Since the angular momentum density along $\hat{z}$ is constant, we can find the initial angular momentum per unit length by multiplying the density by the area πR^2

$$\vec{L}_{EM,init} = \pi R^2 \vec{l}_{EM,init} = \frac{B_{ex}\lambda R^2}{2}\hat{z} \ .$$

(c) The final angular momentum is a combination of the angular momentum stored in the fields and the mechanical angular momentum. To find the angular momentum density stored in the field, we repeat the calculation in (b) but with the field $\vec{B}$ found in (a) replacing $B_{ex}\hat{z}$. We obtain

$$\vec{L}_{EM,f} = \frac{\lambda R^2}{2}\frac{\mu_0 \lambda \omega}{2\pi}\hat{z}.$$

The mechanical angular momentum per unit length is $I\omega\hat{z}$. Thus the final total angular momentum per unit length is

$$\vec{L}_F = \left(\frac{\lambda^2 R^2 \mu_0 \omega}{2\pi} + I\omega\right)\hat{z}.$$

(d) Using conservation of total angular momentum, we set the initial and final angular momenta per unit length equal to each other, i.e., $\vec{L}_{EM,init} = \vec{L}_F$. Solving for ω, we find

$$\omega = \frac{\lambda B_{ex} R^2}{2}\left(\frac{1}{\frac{\mu_0}{4\pi}\lambda^2 R^2 + I}\right).$$

As the initial magnetic field increases, the final angular velocity increases - more angular momentum is stored in the fields.