QUALIFYING EXAMINATION, Part 3

Solutions

Problem 1: Quantum Mechanics I

The term s-wave means $l = 0$ and we can substitute $l = 0$ in the given radial Scrödinger equation.

(a) Inside Solution

In terms of $u(r)$, the Schrödinger equation inside the well is

$$u''(r) + \frac{2\mu}{\hbar^2}(V_0 + E)u(r) = 0\,.$$

Its normalizable solution is

$$u(r) = A\sin Kr\,,$$

where $K^2 = \frac{2\mu}{\hbar^2}(V_0 + E)$, and A is a normalization constant.

(b) Outside Solution

In terms of $u(r)$, the Schrödinger equation outside the well is

$$u''(r) - \left(-\frac{2\mu}{\hbar^2}E\right)u(r) = 0\,.$$

Its normalizable solution is

$$u(r) = Be^{-kr}\,,$$

where $k^2 = -\frac{2\mu}{\hbar^2}E > 0$ and B is a normalization constant.

(c) Matching at the Boundary

Matching the logarithmic derivatives of $u(r)$ at $r = a$ gives

$$\cot Ka = -\frac{k}{K}\,.$$

(d) Solution for $E \ll V_0$

In the limit $E \to 0$, we have $k \to 0$, and the matching condition approaches $\cot\left(\sqrt{\frac{2\mu}{\hbar^2}V_0}\,a\right) = 0$. This equation has solutions

$$\sqrt{\frac{2\mu}{\hbar^2}V_0}\,a = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots\,.$$

The first of these solutions corresponds to a single, zero-energy bound state, approximating the deuteron bound state.

(e) Estimate of V_0

We have

$$V_0 a^2 = \frac{\hbar^2 \pi^2}{8\mu} = 10^{-28}\,\mathrm{MeV} - \mathrm{meter}^2\,,$$

giving $V_0 \simeq 100$ MeV $\gg |E|$.

Problem 2: Quantum Mechanics II

(a) The Hamiltonian $\hat{H}_0$ has orbital SO(3) rotational symmetry, so the energy eigenstates can be grouped into representations of SO(3). The $N = 0$ space has $n_x = n_y = n_z = 0$. Since this level is non-degenerate, it must form the $l = 0$ representation, which is the only one-dimensional representation of SO(3).

We can also write down the ground-state wave function in the coordinate representation and see that it is spherically symmetric (i.e., depends only on r).

(b) the $N = 1$ subspace is spanned by $|1,0,0\rangle\,, |0,1,0\rangle\,, |0,0,1\rangle$. Using the given expression for $\hat{l}_z$ in terms of the raising/lowering operators, we can compute the matrix form of $\hat{l}_z$ in this basis:

$$\hat{l}_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We then find the eigenstates of $\hat{l}_z$ with $m = 0, 1, -1$ to be

$$\begin{aligned} m = 0 &: \ |0,0,1\rangle \\ m = 1 &: \ \frac{1}{\sqrt{2}}(|1,0,0\rangle + i\,|0,1,0\rangle) \\ m = -1 &: \ \frac{1}{\sqrt{2}}(|1,0,0\rangle - i\,|0,1,0\rangle). \end{aligned}$$

(c) $\hat{H}_{\mathrm{SO}}$ is odd under a parity transformation $\hat{I}$, since $\hat{\mathbf{p}} \to -\hat{\mathbf{p}}$ and $\hat{\mathbf{s}} \to \hat{\mathbf{s}}$

$$\hat{I}\hat{H}_{\mathrm{SO}}\hat{I}^{-1} = -\hat{H}_{\mathrm{SO}}.$$

It follows that

$$\begin{aligned} \langle N,l,m,s_z|\hat{H}_{\mathrm{SO}}|N,l,m',s_z'\rangle &= \langle N,l,m,s_z|I^{-1}\cdot I\hat{H}_{\mathrm{SO}}I^{-1}\cdot I|N,l,m',s_z'\rangle \\ &= -(-1)^{2l}\,\langle N,l,m,s_z|\hat{H}_{\mathrm{SO}}|N,l,m',s_z'\rangle \\ &= -\,\langle N,l,m,s_z|\hat{H}_{\mathrm{SO}}|N,l,m',s_z'\rangle\,. \end{aligned}$$

Therefore we must have $\langle N,l,m,s_z|\hat{H}_{\mathrm{SO}}|N,l,m',s_z'\rangle = 0$.

(d) For both $N = 0$ and $N = 1$, the corresponding energy eigenstates of $\hat{H}_0$ have definite orbital angular momentum l ($l = 0$ for $N = 0$ and $l = 1$ for $N = 1$). In degenerate perturbation theory, the first-order correction is obtained by diagonalizing the perturbation in the degenerate energy subspace. Here the matrix elements of the perturbation $\hat{H}_{\rm SO}$ are given by

$$\langle N, l, m, s_z | \hat{H}_{\rm SO} | N, l, m', s_z' \rangle = 0 \ .$$

Thus the eigenstates remain degenerate to first order in γ.

(e) Under time-reversal symmetry $\hat{T}$ we have $\hat{\mathbf{l}} \to -\hat{\mathbf{l}}$ and $\hat{\mathbf{s}} \to -\hat{\mathbf{s}}$. Since $l = 0$ and $s = \frac{1}{2}$, the time-reversal symmetry is implemented by $e^{i\pi s_y} K$ where K is the complex conjugation, and the two $s_z = \pm\frac{1}{2}$ states form a Kramers doublet under $\hat{T}$. Under $\hat{T}$, we have $\hat{\mathbf{p}} \to -\hat{\mathbf{p}}$ and $\hat{\mathbf{s}} \to -\hat{\mathbf{s}}$, so $\hat{H}_{\rm SO}$ respects time-reversal symmetry. Due to Kramers' theorem, this degeneracy cannot be lifted by any perturbation that is invariant under $\hat{T}$.