Problem 1: Statistical Mechanics I

(a) Fermions

\begin{align*}
2\varepsilon & \quad x \quad x \\
\varepsilon & \quad x \\
0 & \\
\end{align*}

(b) Bosons

\begin{align*}
2\varepsilon & \quad x \\
\varepsilon & \quad x \\
0 & \quad x \\
2\varepsilon & \quad x \quad x \\
\varepsilon & \quad x \quad x \\
0 & \quad x \quad x \\
\end{align*}

Figure 1: (a) 3 configurations for 2 fermions, (b) 6 configurations for 2 bosons.

(a) See Fig. 1(a).

(b) See Fig. 1(b).

(c) The canonical partition function for 2 fermions in 3 single-particle levels 0, \(\varepsilon\), 2\(\varepsilon\) is given by

\[ Z = e^{-\beta \varepsilon} + e^{-3\beta \varepsilon} + e^{-2\beta \varepsilon}, \]

and the average energy is

\[
\langle E \rangle = -\frac{d \ln Z}{d \beta} = \frac{\varepsilon e^{-\beta \varepsilon} + 3\varepsilon e^{-3\beta \varepsilon} + 2\varepsilon e^{-2\beta \varepsilon}}{Z}.
\]

(d) The canonical partition function for 2 bosons in 3 single-particle levels 0, \(\varepsilon\), 2\(\varepsilon\) is

\[ Z = e^{-\beta \varepsilon} + e^{-3\beta \varepsilon} + e^{-2\beta \varepsilon} + 1 + e^{-2\beta \varepsilon} + e^{-4\beta \varepsilon}, \]

and the average energy is

\[
\langle E \rangle = -\frac{d \ln Z}{d \beta} = \frac{\varepsilon e^{-\beta \varepsilon} + 3\varepsilon e^{-3\beta \varepsilon} + 2\varepsilon e^{-2\beta \varepsilon} + 0 + 2\varepsilon e^{-2\beta \varepsilon} + 4\varepsilon e^{-4\beta \varepsilon}}{Z}.
\]
(e) For non-interacting fermions, the explicit form of the grand-canonical partition function in Eq. (1) is

\[
Z(\mu, \beta) = 1(N = 0) + e^{\beta \mu} (1 + e^{-\beta \varepsilon} + e^{-2\beta \varepsilon})(N = 1) + e^{2\beta \mu} (e^{-\beta \varepsilon} + e^{-2\beta \varepsilon} + e^{-3\beta \varepsilon})(N = 2) + e^{3\beta \mu} e^{-3\beta \varepsilon}(N = 3).
\]

Treating each of the 3 levels as an independent system of fermions, the partition function is a product of the partition functions of each of the 3 levels. A single level can have an occupation of \(n = 0\) or \(n = 1\). We find

\[
Z(\mu, \beta) = (1 + e^{\beta \mu}) \times (1 + e^{\beta (\mu - \varepsilon)}) \times (1 + e^{\beta (\mu - 2\varepsilon)}).
\]

Opening the parentheses in this equation, we recover the explicit form of Eq. (1) above.

(f) For bosons each level can have occupations of \(n = 0, 1, 2, \ldots\). For the level with energy \(\varepsilon\), we find a partition function of

\[
\sum_{n=0}^{\infty} e^{n \beta (\mu - \varepsilon)} = (1 - e^{\beta (\mu - \varepsilon)})^{-1},
\]

where the geometric series \(\sum_{n=0}^{\infty} e^{n \beta (\mu - \varepsilon)}\) has been summed.

For the 3 level system we find

\[
Z(\mu, \beta) = (1 - e^{\beta \mu})^{-1} \times (1 - e^{\beta (\mu - \varepsilon)})^{-1} \times (1 - e^{\beta (\mu - 2\varepsilon)})^{-1}.
\]

**Problem 2: Statistical Mechanics II**

(a) Using the boundary conditions that the single-particle wave function vanishes on the sides of the cubic box, each state has a volume \((\pi/L)^3\) in momentum space. The states with energy below \(\epsilon = \hbar^2 k^2 / 2m\) occupy a sphere of radius \(k\) in momentum space whose volume is \(\frac{4}{3} \pi k^3\) (the factor of 1/8 is to count only states with positive \(k_i\)). We thus find that the number of states with energy less than \(\epsilon\) is

\[
N_\epsilon = \frac{L^3}{6\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} e^{3/2}. \quad \text{Then } D = dN_\epsilon / d\epsilon \text{ and we find the given result.}
\]

The density of states \(D(\epsilon)\) can also be calculated directly from the density of states in momentum space \(D(k) = (L/\pi)^3\)

\[
\frac{1}{8} \left( \frac{L}{\pi} \right)^3 \frac{d^3 k}{d^3 \vec{k}} = \frac{1}{8} \left( \frac{L}{\pi} \right)^3 4\pi k^2 \frac{dk}{d\epsilon} = \frac{L^3}{2\pi^2} k^2 \frac{d\epsilon}{d\epsilon} \frac{dk}{d\epsilon} = C \sqrt{\epsilon} d\epsilon,
\]

where we have used \(\epsilon = \hbar^2 k^2 / (2m)\) and \(d\epsilon = (\hbar^2 / m) k \, dk\).
(b) In the grand-canonical ensemble, the independent variables are \((\mu_B, \mu_F, T)\). In principle, the volume \(V\) is also one of the independent variables but here we consider it to be constant \(V = L^3\).

The proper potential is the grand (or Landau) potential given by

\[
\Omega = U - TS - \mu_B N_B - \mu_F N_F .
\]

(c) The average number of bosons \(N_B\) is given by

\[
N_B = - \left( \frac{\partial \Omega}{\partial \mu_B} \right)_{T, \mu_F, V} .
\]

The equation for \(N_F\) is obtained by swapping the subscripts \(F\) and \(B\) in the above equation.

(d) Since the bosonic and fermionic subparts are noninteracting, they can be treated independently (at the same temperature \(T\)) and the total partition function is

\[
Z = Z_F(\mu_F, T) Z_B(\mu_B, T) .
\]

The grand potential \(\Omega = -kT \ln Z\) can thus be written as the sum of the grand potentials for the Bose and Fermi gases

\[
\Omega = \Omega_B + \Omega_F .
\]

Using the explicit expression for \(Z_B\) given in the useful formulas, and converting the sum over single-particle states to an integral, we have

\[
\ln Z_B = - \int_0^\infty D(\epsilon) \ln[1 - e^{-\beta(\epsilon - \mu_B)}] \, d\epsilon .
\]

Using the formula for \(N_B\) in part (c) with \(\Omega \rightarrow\Omega_B\) and \(\Omega_B = -kT \ln Z_B\), we find

\[
N_B = \int_0^\infty \frac{D(\epsilon)}{e^{\beta(\epsilon - \mu_B)} - 1} \, d\epsilon .
\]

Carrying out a similar calculation for the fermions, we find

\[
N_F = \int_0^\infty \frac{D(\epsilon)}{e^{\beta(\epsilon - \mu_F)} + 1} \, d\epsilon .
\]

(e) The energy \(U\) is given by \(U = -\partial \ln Z/\partial \beta\). Following a similar method to the calculation in (d), we find

\[
U = U_B + U_F = \int_0^\infty D(\epsilon) \epsilon \left( \frac{1}{e^{\beta(\epsilon - \mu_F)} - 1} + \frac{1}{e^{\beta(\epsilon - \mu_B)} + 1} \right) \, d\epsilon .
\]
(f) At \( T = 0 \), the Fermi-Dirac distribution is a step function at \( \mu(T = 0) = \epsilon_F \). At \( T = 0 \), all bosons occupy the ground state of a single particle in the box of zero energy. Thus

\[
U = \frac{3}{5} N_F \epsilon_F ,
\]

where \( \epsilon_F = C^{-2/3}(3/2)^{2/3}(N_F/L^3)^{2/3} = \frac{\hbar^2}{2m}(6\pi^2 n_F)^{2/3} \) and \( n_F = N_F/L^3 \).

The state of the system is a mixture of a Fermi sea (filled up to energy \( \mu_F = \epsilon_F \)) and a pure Bose-Einstein condensate (with \( \mu_B = 0 \)).

(g) At a very low temperature, only the lowest excited many-particle states contribute (in addition to the ground state). Such excited states are sketched in the figure for both the Bose and Fermi gases.