QUALIFYING EXAMINATION, Part 1

Solutions

Problem 1: Classical Mechanics I

(a)

$$\begin{aligned} T &= \frac{M}{2}\dot{x}_1^2 + \frac{m}{2}((\dot{x}_1 + \dot{x}_2\cos\alpha)^2 + \dot{x}_2^2\sin^2\alpha) \\ &= \frac{M}{2}\dot{x}_1^2 + \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2 + 2\dot{x}_1\dot{x}_2\cos\alpha) \\ V &= -mgx_2\sin\alpha \\ L &= T - V \end{aligned}$$

(b)

$$\begin{aligned} p_1 &= \frac{\partial L}{\partial \dot{x}_1} = M\dot{x}_1 + m(\dot{x}_1 + \dot{x}_2\cos\alpha) \\ p_2 &= \frac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2 + m\dot{x}_1\cos\alpha \end{aligned}$$

(c)

$$\begin{aligned} \dot{p}_1 &= \frac{\partial L}{\partial x_1} = 0 \\ \dot{p}_2 &= \frac{\partial L}{\partial x_2} = mg\sin\alpha \end{aligned}$$

(d) p_1 is the conserved momentum of the center of mass in the horizontal direction.

(e) Combining

$$(m+M)\ddot{x}_1 + m\ddot{x}_2\cos\alpha = 0$$

and

$$m\ddot{x}_2 + m\ddot{x}_1\cos\alpha = mg\sin\alpha,$$

we find

$$\ddot{x}_2 = \frac{g\sin\alpha(m+M)}{M + m\sin^2\alpha}.$$

Setting

$$\frac{1}{2}\ddot{x}_2 t^{*2} = \frac{h}{\sin\alpha}, \text{ the distance down the slope,}$$

we find

$$t^{*2} = \frac{2h(M + m\sin^2\alpha)}{g\sin^2\alpha(m+M)}.$$

(f) We find the location of P either by using the analogous equation for $\ddot{x}_1$ or the fact that the center of mass cannot move. So the change in the CM coordinate, which is the weighted sum of the changes in x_1 and horizontal shift in x_2, must also be 0, which means

$$0 = Mx_1(t^*) + m(x_1(t^*) + h \cot \alpha),$$

leading to

$$x_1(t^*) = -\frac{mh}{(M+m)} \cot \alpha.$$

Problem 2: Classical Mechanics II

(a) In the clock frame $\Delta x' = 0$, and it follows from the Lorentz transformation that $\Delta t = \gamma \Delta t'$. Thus, when the time is $t' = 1s$ in the clock's frame, it will be $t = 1/\sqrt{1-(9/25)} = \frac{5}{4}s$ in my frame. The clock will then be at $x = vt = \frac{3c}{5} \times \frac{5}{4}s = 0.75$ light-seconds $= 0.75 \times 3 \cdot 10^8 \frac{m}{s} = 2.25 \cdot 10^8 m$ in my frame.

(b) In my frame, the light pulse travels a distance of $\Delta x = 0.75\, c$ at speed c and will need $\Delta t = \Delta x / c = 0.75\, s$ to reach me. Since the pulse was sent at $t = \frac{5}{4}s$, it will reach me when my time is $t = \frac{5}{4}s + \frac{3}{4}\, s = 2\, s$.

(c) In my frame the two events occur at

$$x_A = t_A = 0, \quad x_B = 3 \cdot 10^3 m, \quad t_B = 4 \cdot 10^{-6} s\ .$$

In the frame in which the two events occur simultaneously

$$\Delta t' = \frac{\Delta t - v \Delta x / c^2}{\sqrt{1 - u^2/c^2}} = 0 \rightarrow \frac{v}{c} = 0.4\ .$$

(d) Yes, because they cannot be causally connected: even a light pulse (and therefore any signal) traveling at a speed $3 \cdot 10^8 \frac{m}{s}$ for $4 \cdot 10^{-6} s$ cannot go from A to B which are $3 \cdot 10^3 m$ apart.

We can also see from the above equation for $\Delta t'$ that $\Delta t' < 0$ when $v > 0.4\, c$.

(e) In the frame in which the events occur at the same point $\Delta x' = 0$. Using

$$\Delta x' = \gamma(\Delta x - v \Delta t) = 0$$

gives $v = \frac{\Delta x}{\Delta t} = \frac{3 \cdot 10^3 m}{4 \cdot 10^{-6} s} = 7.5 \cdot 10^8\, m/s$, which is greater than the speed of light c. Therefore, there is no frame in which the events occur at the same point.

Equivalently, if a light pulse cannot be at both events (as we have seen), neither can any observer.