QUALIFYING EXAMINATION, Part 2

Solutions

Problem 1: Electromagnetism I

(a) The charge density is given by

$$\rho(x, y, z) = e[\delta(z-a) + \delta(z+a) - 2\delta(z)]\delta(x)\delta(y)$$

The total charge Q is

$$Q = \int \rho(x')dV' = e\int [\delta(z'+a) + \delta(z'-a) - 2\delta(z')]dz' = e + e - 2e = 0\ .$$

The dipole moment P_i is calculated using

$$P_i = \int \rho(x')x_i dV'\ .$$

We find

$$P_x = P_y = 0\ .$$

and

$$P_z = e\int [\delta(z'-a) + \delta(z'+a) - 2\delta(z')]z'dz' = e(a - a + 0) = 0\ .$$

The quadrupole moment Q_{ij} is calculated using

$$Q_{ij} = \int \rho(3x'_i x'_j - \delta_{ij}r'^2)dV'\ .$$

We find

$$Q_{xx} = \int \rho(3x'^2 - r'^2)dV' = -\int \rho(z')z'^2 dz' = -e(a^2 + a^2 - 0) = -2ea^2\ .$$

Similarly,

$$Q_{yy} = -2ea^2$$

.

and

$$Q_{zz} = \int \rho(3z'^2 - r'^2)dV' = 2\int \rho(z')z'^2 dz' = 4ea^2 \ .$$

The off-diagonal terms are all zero. Thus the contribution of the quadrupole moment to the potential is

$$\begin{aligned}
\Phi(\mathbf{x}) &= \frac{Q_{ij}x_i x_j}{2r^5} = \frac{1}{2r^5}(Q_{xx}x^2 + Q_{yy}y^2 + Q_{zz}z^2) \\
&= \frac{2ea^2}{2r^5}(2z^2 - (x^2+y^2)) = \frac{ea^2}{r^3}(2\cos^2\theta - \sin^2\theta) = \frac{ea^2}{r^3}(3\cos^2\theta - 1) \ .
\end{aligned}$$

(b) We can rewrite the expression (b) as

$$\Phi(\mathbf{x}) = \frac{ea^2}{r^3} 2P_2(\cos\theta) \ ,$$

where $P_2(x) = \frac{1}{2}(3x^2 - 1)$.

(c) The electric field is given by $\vec{E} = -\vec{\nabla}\Phi$. Using the expression for $\vec{\nabla}$ in spherical coordinates and the result for Φ in part (b), we find

$$\vec{E} = \frac{3ea^2}{r^4}\left[(3\cos^2\theta - 1)\hat{r} + 2\cos\theta\sin\theta\ \hat{\theta}\right] \ .$$

(d) At a point on the positive z axis at a distance $r > a$ from the origin, the exact electric field is given by

$$\vec{E}(r) = \left[\frac{e}{(r-a)^2} + \frac{e}{(r+a)^2} - \frac{2e}{r^2}\right]\hat{r} = \left[2e\frac{r^2+a^2}{(r^2-a^2)^2} - \frac{2e}{r^2}\right]\hat{r} \ .$$

or

$$\vec{E}(r) = \frac{2e}{r^2}\left[\frac{1+(a/r)^2}{[1-(a/r)^2]^2} - 1\right]\hat{r} \ .$$

Using $(1-\epsilon)^{-1} = 1 + \epsilon + O(\epsilon^2)$, we find to leading order

$$E(r) \approx \frac{6ea^2}{r^4}\hat{r} \ .$$

This coincides with the result in (c) when taking $\theta = 0$.

Problem 2: Electromagnetism II

(a) We can compute the Fourier transform as

$$\tilde{\vec{A}}(\vec{x},\omega) = \frac{1}{2\pi}\int dt e^{i\omega t}\int d^3x' \frac{\vec{J}\left(\vec{x}', t-\frac{|\vec{x}-\vec{x}'|}{c}\right)}{c|\vec{x}-\vec{x}'|} = \int d^3x' \frac{1}{2\pi}\int dt' e^{i\omega(t'+\frac{|\vec{x}-\vec{x}'|}{c})} \frac{\vec{J}(\vec{x}',t')}{c|\vec{x}-\vec{x}'|}$$

$$= \int d^3x' \frac{\tilde{\vec{J}}(\vec{x}',\omega)e^{i\frac{\omega}{c}|\vec{x}-\vec{x}'|}}{c|\vec{x}-\vec{x}'|}.$$

A similar calculation gives

$$\tilde{\phi}(\vec{x},\omega) = \int d^3x' \frac{\tilde{\rho}(\vec{x}',\omega)e^{i\frac{\omega}{c}|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}.$$

Since the oscillations occur at frequency ω, the conservation relation takes the form

$$\vec{\nabla}\cdot\tilde{\vec{J}}(\vec{x},t) + \frac{\partial}{\partial t}\tilde{\rho}(\vec{x},t) = 0 \to \vec{\nabla}\cdot\tilde{\vec{J}}(\vec{x},\omega) - i\omega\ \tilde{\rho}(\vec{x},\omega) = 0.$$

(b) In the limit $r \gg \lambda \gg d$, we can approximate $|\vec{x}-\vec{x}'| \approx r$ and take it outside of the integral, giving

$$\tilde{\vec{A}}(\vec{x},\omega) = \frac{e^{i\frac{\omega}{c}r}}{cr} \int d^3x' \vec{J}\left(\vec{x}',\omega\right) = \frac{e^{2\pi i\frac{r}{\lambda}}}{cr} \int d^3x' \vec{J}\left(\vec{x}',\omega\right).$$

The result falls off linearly $\sim 1/r$ with a rapidly oscillating phase.

(c) Let us use the given identity

$$0 = \int d^3x'\ \vec{\nabla}'\cdot\left[x_i'\ \tilde{\vec{J}}(\vec{x}',\omega)\right] = \int d^3x'\left[\tilde{J}_i(\vec{x}',\omega) + x_i'\vec{\nabla}'\cdot\tilde{\vec{J}}(\vec{x}',\omega)\right]$$

$$= \int d^3x'\left[\tilde{J}_i(\vec{x}',\omega) + i\omega x_i'\tilde{\rho}(\vec{x}',\omega)\right],$$

or equivalently

$$\frac{e^{i\frac{\omega}{c}r}}{cr}\int d^3x'\tilde{J}_i(\vec{x}',\omega) = -i\frac{\omega}{c}\frac{e^{i\frac{\omega}{c}r}}{r}\int d^3x' x_i'\tilde{\rho}(\vec{x}',\omega).$$

So after substituting our formula for $\tilde{\vec{A}}$ we conclude that

$$\tilde{\vec{A}}(\vec{x},\omega) = -i\frac{\omega}{c}\ \frac{e^{i\frac{\omega}{c}r}}{r}\ \tilde{\vec{p}}(\omega).$$

(d) To obtain the magnetic field we take the curl of the vector potential

$$\tilde{\vec{B}}(\vec{x},\omega) = \vec{\nabla}\times\tilde{\vec{A}}(\vec{x},\omega) = \hat{r}\frac{\partial}{\partial r}\times\tilde{\vec{A}}(\vec{x},\omega) \simeq \left(\frac{\omega}{c}\right)^2\frac{e^{i\frac{\omega}{c}r}}{r}\hat{r}\times\tilde{\vec{p}}(\omega),$$

where we dropped the subleading term which falls off as $\sim 1/r^2$.

The leading electric field can be readily obtained using Ampére's law in vacuum

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \to \hat{r}\frac{\partial}{\partial r}\times\tilde{\vec{B}}(\vec{x},\omega) \simeq \left(i\frac{\omega}{c}\right)\hat{r}\times\tilde{\vec{B}}(\vec{x},\omega) = \frac{1}{c}(-i\omega)\tilde{\vec{E}}(\vec{x},\omega),$$

so we conclude that

$$\tilde{\vec{E}}(\vec{x},\omega) = \tilde{\vec{B}}(\vec{x},\omega)\times\hat{r}.$$

(e) We can Fourier transform back using

$\vec{A}(\vec{x},t) = \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \tilde{\vec{A}}(\vec{x},\omega) = \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \left(-i\frac{\omega}{c}\, \frac{e^{i\frac{\omega}{c}r}}{r} \int d^3x' \vec{x}' \rho(\vec{x}') \delta(\omega - \omega_0) \right)$

$= -i\frac{\omega_0}{c}\, \frac{e^{-i\omega_0 (t - \frac{r}{c})}}{r}\, \vec{p},$

where $\vec{p} = \int d^3x'\ \rho(\vec{x}')\, \vec{x}'$.