QUALIFYING EXAMINATION, Part 3

Solutions

Problem 1: Quantum Mechanics I

(a) Using the expansion of $|\psi\rangle$, the RHS of the Schrödinger equation is

$$H(t)\,|\psi(t)\rangle \;=\; \sum_n c_n(t)e^{i\theta_n(t)}H(t)\,|n(t)\rangle = \sum_n c_n(t)e^{i\theta_n(t)}E_n(t)\,|n(t)\rangle \;\;.$$

The LHS of the Schrödinger equation is

$$\begin{aligned} i\,|\dot\psi(t)\rangle \;&=\; i\sum_n \left(\dot c_n e^{i\theta_n}\,|n\rangle + ic_n\dot\theta_n e^{i\theta_n}\,|n\rangle + c_n e^{i\theta_n}\,|\dot n\rangle\right) \\ &=\; i\sum_n \left(\dot c_n e^{i\theta_n}\,|n\rangle - ic_n E_n e^{i\theta_n}\,|n\rangle + c_n e^{i\theta_n}\,|\dot n\rangle\right) \;\;, \end{aligned}$$

where in the second line we used $\dot\theta_n = -E_n$ (this follows directly from the definition of θ_n and the fundamental theorem of calculus).

The middle term of this expression is equal to the RHS of the Schrödinger equation, so we get the desired euqation:

$$0 \;=\; \sum_n e^{i\theta_n}\left(\dot c_n\,|n\rangle + c_n\,|\dot n\rangle\right) \;\;.$$

(b) Take the result of (a) and act on it (from the left) with $\langle k|$:

$$\begin{aligned} 0 \;&=\; \langle k|\left[\sum_n e^{i\theta_n}\left(\dot c_n\,|n\rangle + c_n\,|\dot n\rangle\right)\right] = \sum_n e^{i\theta_n}\left(\dot c_n\,\langle k|n\rangle + c_n\,\langle k|\dot n\rangle\right) = \sum_n e^{i\theta_n}\left(\dot c_n\delta_{k,n} + c_n\,\langle k|\dot n\rangle\right) \\ &=\; \dot c_k e^{i\theta_k} + \sum_n e^{i\theta_n}c_n\,\langle k|\dot n\rangle \end{aligned}$$

Solving for $\dot c_k$ gives the final result:

$$\dot c_k \;=\; -\sum_n e^{i(\theta_n-\theta_k)}c_n\,\langle k|\dot n\rangle \;\;.$$

(c) To find an expression for $\langle k|\dot n\rangle$, we take a time derivative of Eq. (1) in the problem:

$$\dot H\,|n\rangle + H\,|\dot n\rangle \;=\; \dot E_n\,|n\rangle + E_n\,|\dot n\rangle \;\;.$$

Then take the inner product (from the left) with $\langle k|$ and use $\langle k|\,H = \langle k|\,E_k$ to obtain

$$\langle k|\dot H|n\rangle + E_k\,\langle k|\dot n\rangle \;=\; \dot E_n\,\langle k|n\rangle + E_n\,\langle k|\dot n\rangle = E_n\,\langle k|\dot n\rangle \;\;.$$

Solving for $\langle k|\dot{n}\rangle$ gives

$$\langle k|\dot{n}\rangle \quad = \quad \frac{\langle k|\dot{H}|n\rangle}{E_n - E_k} \; .$$

(d) By normalization, $\langle n|n\rangle = 1$. Taking a time derivative gives

$$0 \quad = \quad \frac{d}{dt}\,\langle n|n\rangle = \langle n|\dot{n}\rangle + \langle \dot{n}|n\rangle = \langle n|\dot{n}\rangle + (\langle n|\dot{n}\rangle)^* = 2\,\mathrm{Re}(\langle n|\dot{n}\rangle) \; .$$

Thus the real part of $\langle n|\dot{n}\rangle$ vanishes and it is purely imaginary.

Problem 2: Quantum Mechanics II

(a) The potential is spherically symmetric and therefore H is invariant under rotations. It follows that the angular momentum $\vec{L}$ is conserved as the generator of rotations. Thus $[L_i, H] = 0$.

(b) We have

$$[(\vec{p} \times \vec{L})_i]^\dagger = \epsilon_{ijk}(p_j L_k)^\dagger = \epsilon_{ijk} L_k^\dagger p_j^\dagger = \epsilon_{ijk} L_k p_j = -\epsilon_{ijk} L_j p_k = -(\vec{L} \times \vec{p})_i \, .$$

The combination $\vec{p} \times \vec{L} - \vec{L} \times \vec{p}$ is therefore Hermitian, and so is the Laplace-Runge-Lenz vector $\vec{A}$.

(c) As we know from the study of angular momentum, t must be an integer or half integer, $t = 0, 1/2, 1, \cdots$, and both m_T and m_S take values in integer steps between $-t$ and t, namely $m_T, m_S = -t, -t+1, \ldots, t-1, t$.

(d) Using Eq. (2) in the problem and equating $\vec{T}^2$ to its eigenvalue $t(t+1)$, we find

$$t(t+1) = -\frac{m_e e^4}{8E} - \frac{1}{4} \, .$$

Solving for E, using $4t(t+1) + 1 = (2t+1)^2$, we obtain

$$E = -\frac{m_e e^4}{2(2t+1)^2} \, .$$

Defining $n = 2t + 1$, we can write these energy eigenvalues in the form

$$E = -\frac{m_e e^4}{2n^2} \, .$$

Since t is integer or half integer, the allowed values of n are $n = 1, 2, \ldots$.

(e) Since each of the m_T and m_S can have independently $2t + 1$ values, the degeneracy of level n is $(2t + 1)^2 = n^2$.

(f) Since $\vec{L} = \vec{T} + \vec{S}$ we can use the addition rule of two angular momenta t to find that the allowed values of l (for a given n) range between 0 and $2t$ in steps of 1. Since $2t$ is always an integer, we have $l = 0, 1, \ldots, 2t$ or

$$l = 0, 1, \ldots, n-1 \, ,$$

which is the known result for the hydrogen atom. The total degeneracy of level n is given by

$$\sum_{l=0}^{n-1} (2l + 1) = n^2 \, .$$

which is the same as the result in (e).