QUALIFYING EXAMINATION, Part 4

Solutions

Problem 1: Statistical Mechanics I

(a) The classical single-particle partition Z_1 is given by

$$\begin{aligned} Z_1 &= \int \frac{d^3\vec{p}\,d^3\vec{r}}{h^3} \exp\left[-\beta\left(\frac{\vec{p}^2}{2m} + mgz\right)\right] \\ &= \frac{1}{h^3}\left[\int d^3\vec{p} \exp\left(-\beta\frac{\vec{p}^2}{2m}\right)\right]\left[\int dz\, A\, \exp(-\beta mgz)\right] \\ &= \left(\frac{1}{h}\right)^3 \times \left(\frac{2\pi m}{\beta}\right)^{3/2} \times \left(\frac{A}{\beta mg}\right) . \end{aligned}$$

The partition function for the N distinguishable non-interacting particles is given by

$$Z_N = Z_1^N = \left(\frac{1}{h}\right)^{3N} \times \left(\frac{2\pi m}{\beta}\right)^{3N/2} \times \left(\frac{A}{\beta mg}\right)^N .$$

(b) The expectation value of z is given by

$$\langle z \rangle = \int d^3\vec{p}\,d^3\vec{r}\, z\, \rho(\vec{p},\vec{r}) ,$$

where $\rho(\vec{p},\vec{r}) = e^{-\beta E}/Z_1$ is the classical density distribution. We find

$$\begin{aligned} \langle z \rangle &= \int d^3\vec{p}\,d^3\vec{r}\, z\, \exp\left[-\beta\left(\frac{\vec{p}^2}{2m} + mgz\right)\right] /Z_1 \\ &= -\frac{1}{\beta m} Z_1^{-1} \frac{\partial}{\partial g} \int d^3\vec{p}\,d^3\vec{r} \exp\left[-\beta\left(\frac{\vec{p}^2}{2m} + mgz\right)\right] \\ &= -\frac{1}{\beta m}\frac{\partial}{\partial g} \ln Z_1 . \end{aligned}$$

Using the expression above for Z_1, we find

$$\langle z \rangle = \frac{1}{\beta mg} .$$

(c) Taking a thin layer dz at height z, we have from the balance of forces (pressure and gravity)

$$[P(z+dz) - P(z)]A = -Nmg\rho(z)dz ,$$

where $P(z)$ is the pressure at height z and

$$\rho(z) = \beta mg e^{-\beta mgz}$$

is the probability density to find a gas particle at height z. We find

$$\frac{dP}{dz} = -\frac{Nmg}{A}\beta mg e^{-\beta mgz} \ .$$

Integrating from z to infinity and using $P(z=\infty)=0$, we find

$$P(z) = \frac{Nmg}{A}\exp(-\beta mgz) \ .$$

This result can also be found from the total average weight of the column of gas particles above height z.

(d) Using the result in (c), the pressure at $z=0$ is Nmg/A and the force is Nmg. This is just the total weight of all N gas particles.

(e) At $z=0$ the pressure is independent of temperature, and at all $z>0$ it is an increasing function of temperature.

(f) At the surface of the earth ($z=0$), the pressure should not depend much on temperature, but at high altitude the pressure will be lower on cold days, at least in this approximation.

Problem 2: Statistical Mechanics II

(a) At $T = 0$ the state of the system is two Fermi seas of spins $\uparrow$ and $\downarrow$ filling all single-particle states up to energies $\epsilon_{F\uparrow}$ and $\epsilon_{F\downarrow}$ respectively. The total occupation number is:

$$\rho_{\text{tot}}(\epsilon) = \Theta(\epsilon_{F\uparrow} - \epsilon) + \Theta(\epsilon_{F\downarrow} - \epsilon) \ ,$$

where Θ is the Heaviside step function.

(b) We have $N_\sigma = \sum_i \Theta(\mu_\sigma - \epsilon_i)$ where the sum i extends to all single-particle states of the corresponding spin species. Taking the continuum limit, the sum is replaced by an integral over energy, introducing the density of states $\mathcal{D}(\epsilon)$:

$$n_\sigma = \frac{1}{V}\int_0^{\epsilon_{F\sigma}} \mathcal{D}(\epsilon)\mathrm{d}\epsilon \ .$$

Calculating this integral explicitly, we find $\epsilon_{F\sigma} = \alpha n_\sigma^{2/3}$, where $\alpha = \frac{\hbar^2}{2m}(6\pi^2)^{2/3}$.

(c) For each spin species we have

$$\frac{U_{\text{kin},\sigma}}{N_\sigma} = \frac{\int_0^{\epsilon_{F\sigma}} \mathcal{D}(\epsilon)\epsilon\mathrm{d}\epsilon}{\int_0^{\epsilon_{F\sigma}} \mathcal{D}(\epsilon)\mathrm{d}\epsilon} = \frac{3}{5}\epsilon_{F\sigma}.$$

(d) Combining the results of parts (b) and (c), the total kinetic energy per unit volume is

$$\begin{aligned}\sum_{\sigma=\uparrow,\downarrow} U_{\text{kin},\sigma}/V = \frac{3}{5}\alpha \sum_{\sigma=\uparrow,\downarrow} n_\sigma^{5/3} &= \frac{3}{5}\alpha\left(\frac{n}{2}\right)^{5/3}\left[(1+\delta)^{5/3} + (1-\delta)^{5/3}\right] \\ &\approx \frac{3}{5}\alpha\left(\frac{n}{2}\right)^{5/3}\left(2 + \frac{10}{9}\delta^2\right).\end{aligned}$$

(e) The interaction energy per unit volume is

$$\frac{U_{\text{int}}}{V} = g\left(\frac{n}{2}\right)^2(1-\delta^2) \ .$$

(f) For the total energy per unit volume, written as $u_{\text{tot}}(\delta) = c_0 + c_2\delta^2 + O(\delta^4)$, we find:

$$c_0 = \frac{6}{5}\alpha\left(\frac{n}{2}\right)^{5/3} + g\left(\frac{n}{2}\right)^2 \ , \qquad c_2 = \frac{2}{3}\alpha\left(\frac{n}{2}\right)^{5/3} - g\left(\frac{n}{2}\right)^2 \ .$$

The total kinetic energy and potential energy are even functions of δ and thus all odd coefficients of the expansion in δ vanish.

(g) We see that if $g > 0$, then depending on its value, c_2 changes sign. This occurs at a critical interaction strength of

$$g_{\rm c} = \frac{2}{3}\alpha\left(\frac{n}{2}\right)^{-1/3}.$$

(h) The magnetization $M \propto \delta$. For $g < g_c$, we have $c_2 > 0$ and the total energy is minimized for $\delta = 0$ (or $M = 0$), the paramagnetic state. For $g > g_c$, we have $c_2 < 0$ for which $\delta = 0$ is a local maximum and the energy is minimized at $\pm\delta \neq 0$, i.e., the system becomes *polarized* as $M \neq 0$. Plotting the minima of M versus g, one finds a bifurcation, where $M = 0$ for $g < g_{\rm c}$ and there are two symmetric (non-zero) branches for M when $g > g_{\rm c}$ (the sign of M is not determined *a priori*, an example of spontaneous symmetry breaking).

In fact, we can be more quantitative (not required in the problem): we find the minimum of M by solving $\partial u_{\rm tot}/\partial\delta = 0$. Close enough to the transition (from above), we can keep only the lowest nontrivial terms (here, up to c_4). We find $\delta^2 = -c_2/(2c_4)$. Since $c_4 > 0$ this has nontrivial solutions provided $g > g_{\rm c}$ (for which $c_2 < 0$). We find $M \propto \pm\sqrt{g - g_{\rm c}}$. For $g > g_{\rm c}$, the paramagnetic state becomes unstable, and the system develops spontaneous magnetization, i.e., it becomes ferromagnetic. That's the Stoner ferromagnetic phase transition.