The Thermal Physics of "Empty" Space:

Heating the Vacuum with Heavy Ions

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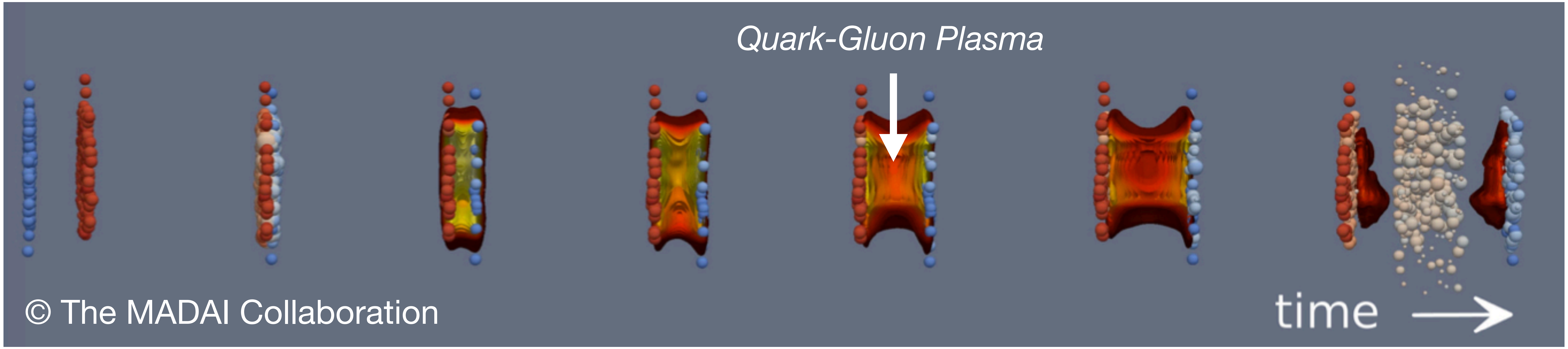
Yale University

Introduction: Thermalization

Thermalization in nuclear collisions

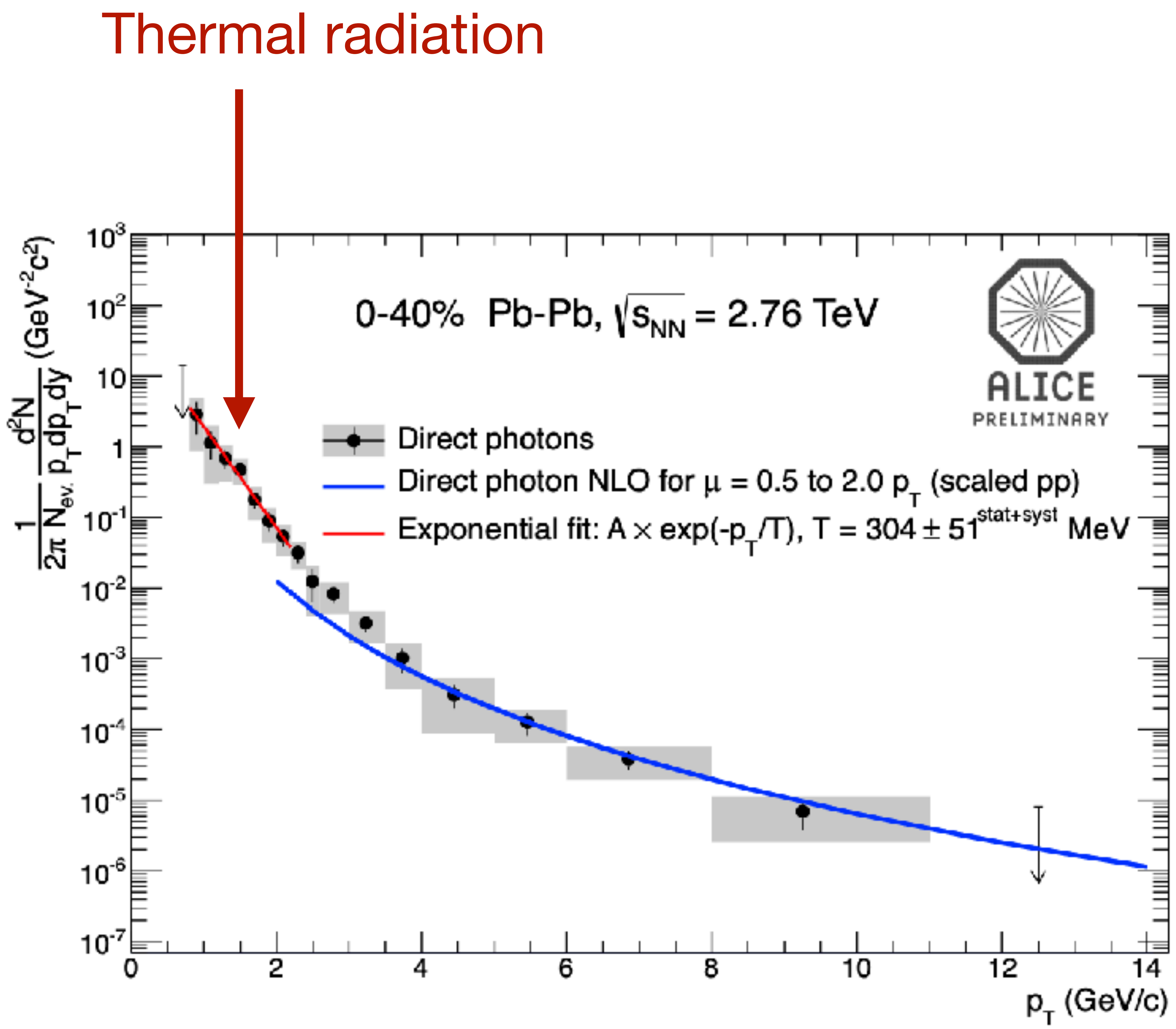


- High-energy heavy ion collisions are studied because they convert a large amount of kinetic energy into heat.
- At high energy, the nuclei pass through each other and leave between them a region of "hot" space with the quantum numbers of the vacuum.

- The result is the formation of a state of matter called *Quark-Gluon Plasma*, whose creation requires $T > 2{\times}10^{12}$ K.
- How long does that take for "empty" space to thermalize?
- How thermal is this "fireball"?
- How does the thermalization process work?

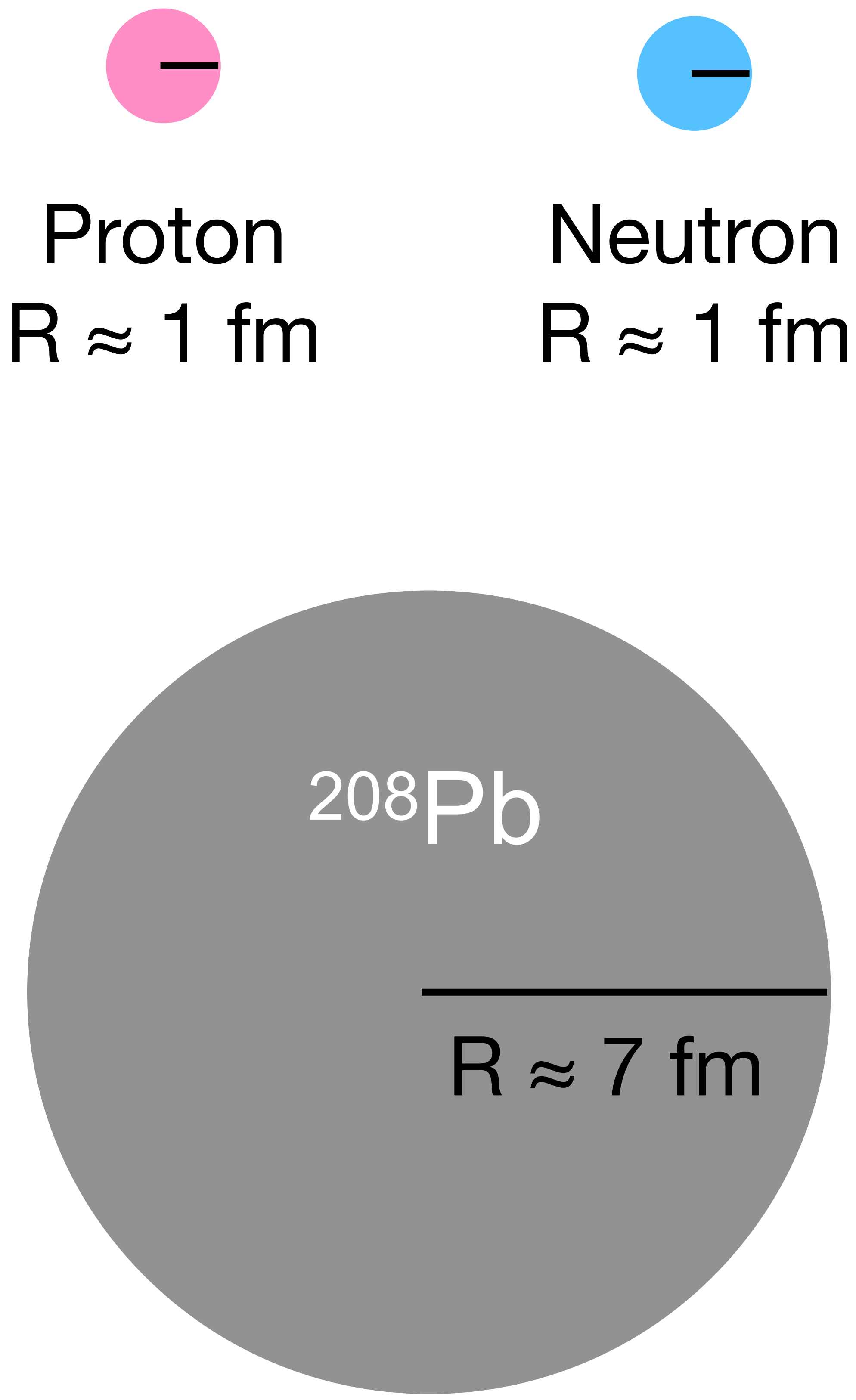
Some facts and conventions

- The SI unit of temperature T is degrees Kelvin (K).
- In this talk I will use instead the thermal energy $k_B T$ and simply call it "T".
- Convenient energy unit: 10^6 electron volt (MeV).
- $(k_B)T$ = 1 MeV corresponds to $T = 1.16\times10^{10}$ K.
- $(k_B)T$ = 100 MeV corresponds to $T \approx$ 1 trillion K.
- The core of the sun is at $T \approx 0.0015$ MeV.
- The highest temperature that has been directly measured using blackbody radiation from a nuclear collision is $T \approx 300$ MeV.
- This is a time average - collision models indicate that the initial temperature was $T_i \approx 500$ MeV and was reached $\approx 2\times10^{-24}$ s after the nuclei collided.



Some more facts and conventions

- Convenient length scale: 1 femtometer (fm) = 10^{-15} m. This is approximately the charge radius of a proton.
- The unit “fm” is often called “Fermi”.
- A large nucleus, such as ^{208}Pb, has a radius of ≈ 7 fm.
- Convenient unit of time: The time it takes for light to travel 1 fm:
  - 1 fm/c = 3.3×10^{-24} s.
- Quantum physics connects time/space and energy units as follows:
  - $\hbar c$ / (200 MeV) ≈ 1 fm
  - $\hbar$ / (200 MeV) ≈ 1 fm/c
- The factors $\hbar$ and c are often omitted (“natural units”)

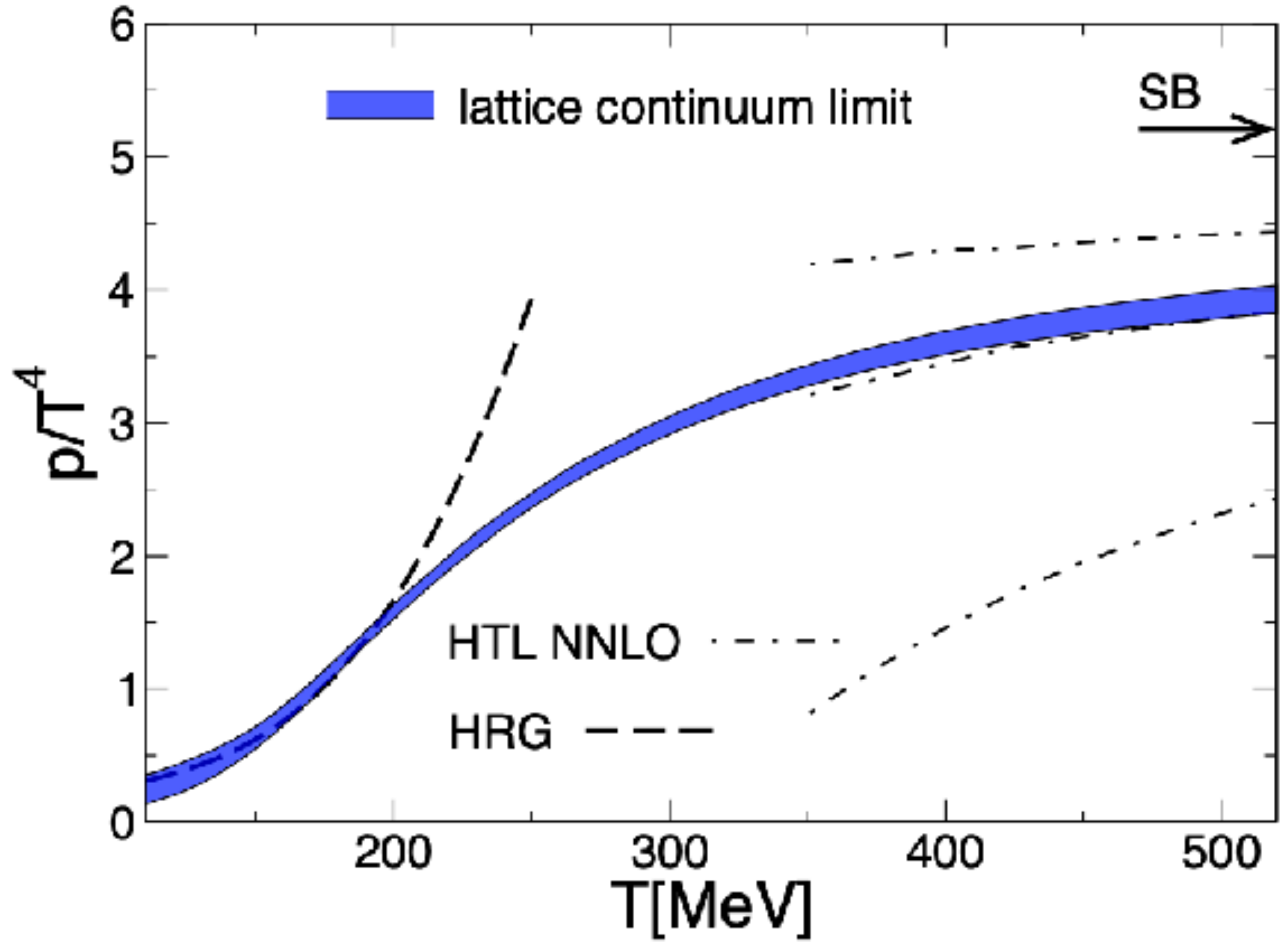


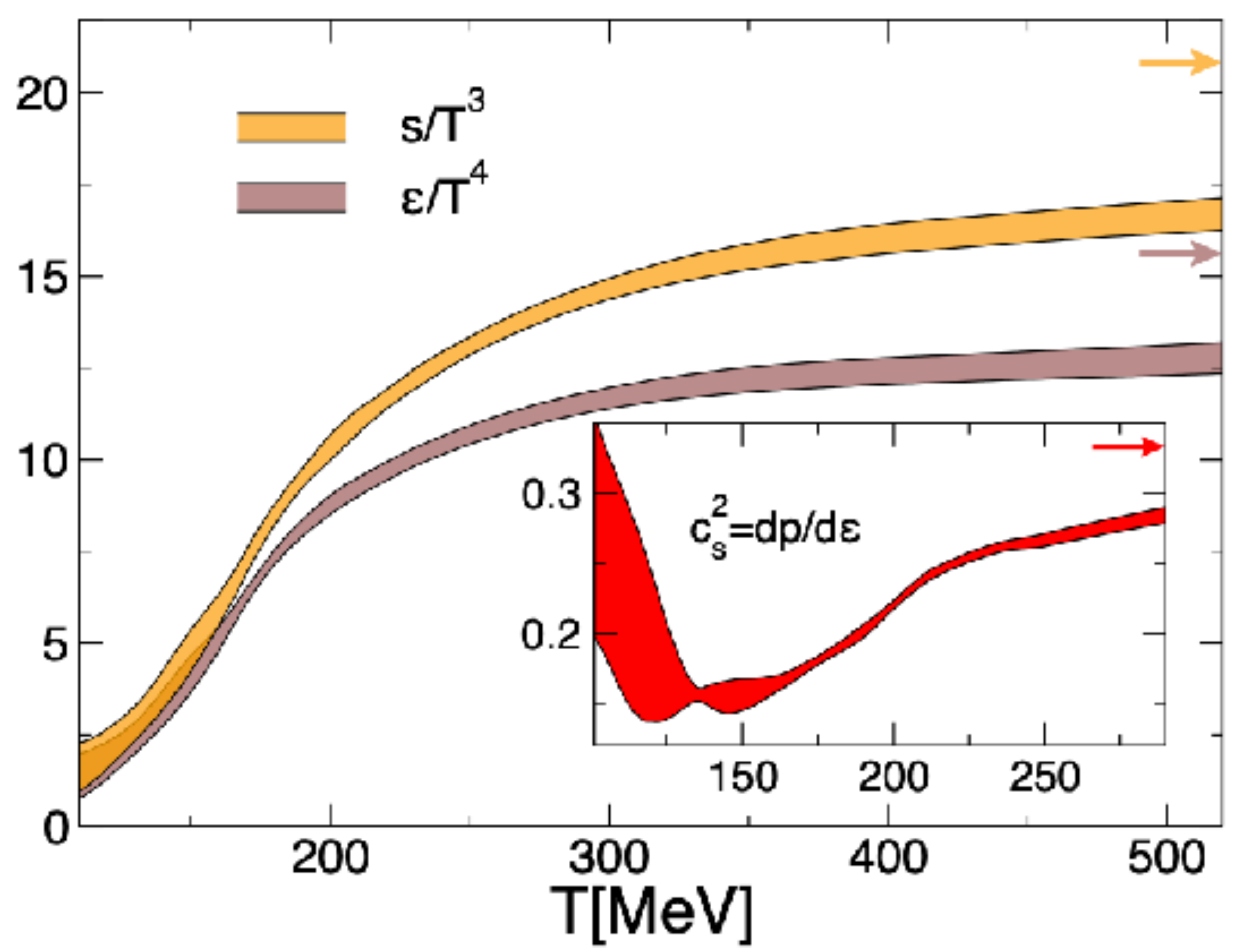
Thermal QCD

Thermal QCD equation of state is known with precision from lattice

BUT

Evidence for thermal behavior must come from experiment



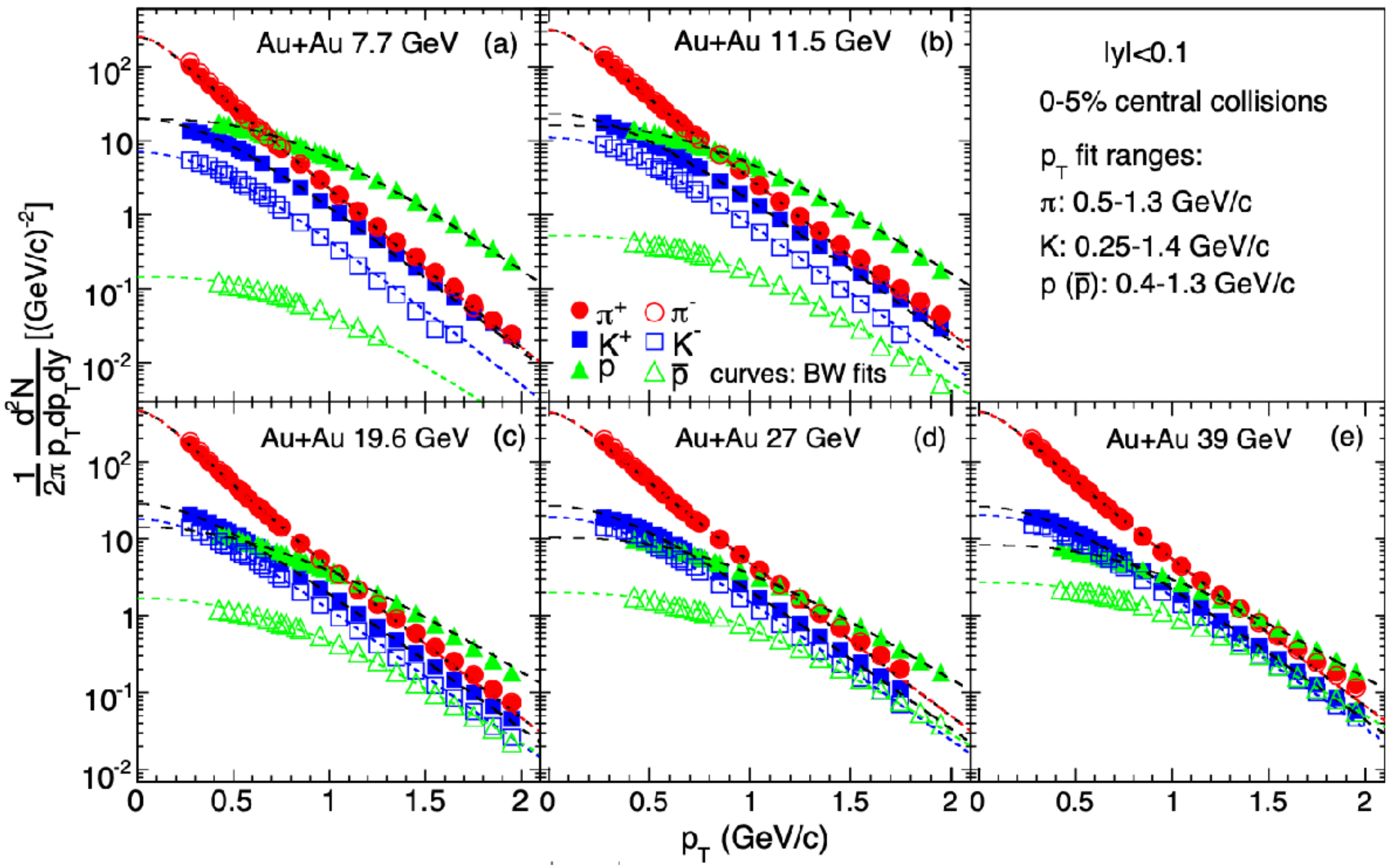


Evidence for the thermalization of matter in a heavy ion collision comes from multiple sources:

- Spectra: Thermal radiation from a collectively moving source.
- Particle yields: Emission from a common thermal system.
- Flow pattern: Relativistic viscous hydrodynamics.
- *n*-particle correlations: Common initial density fluctuations.

Evidence for thermalization: Spectra

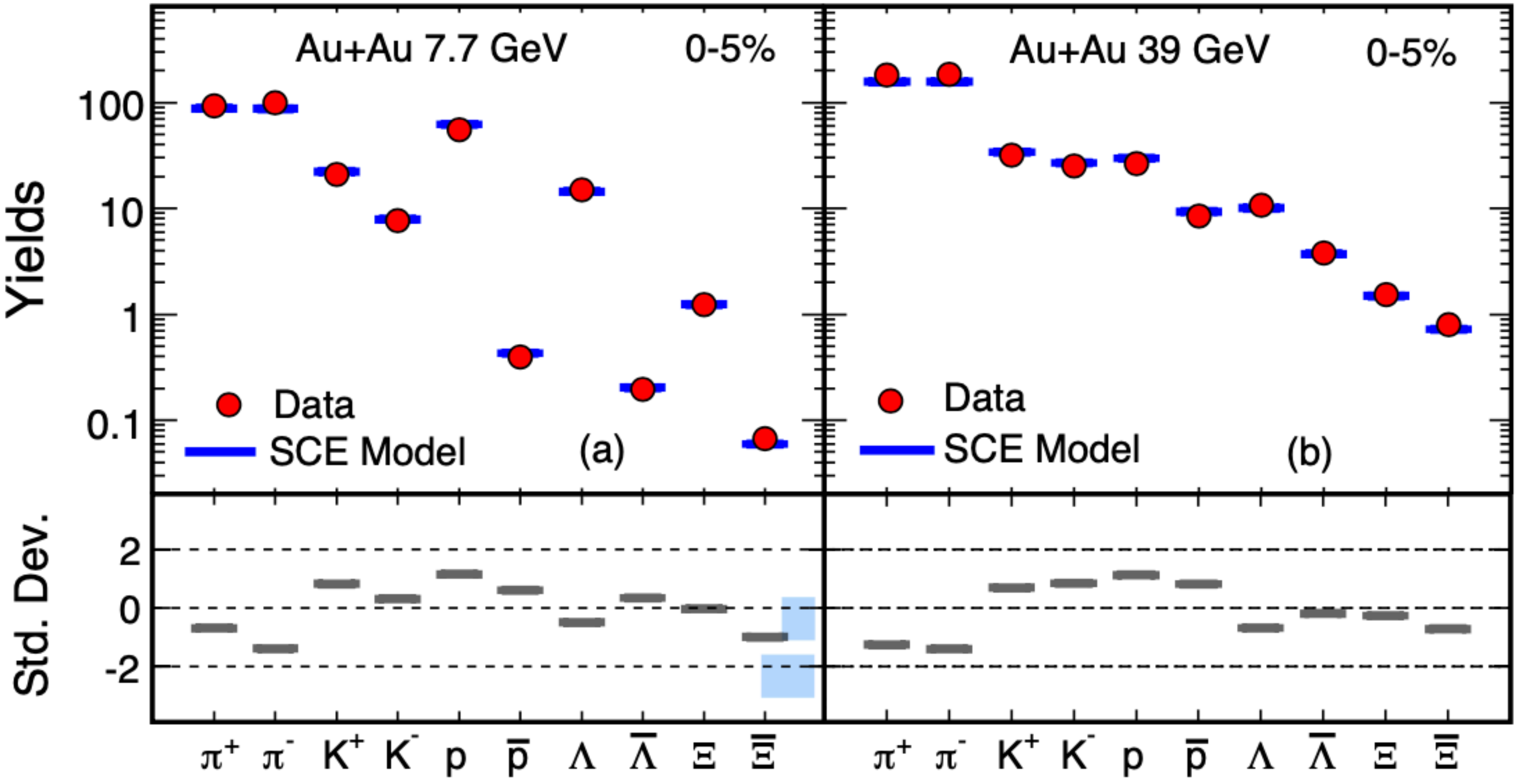
STAR: *Phys. Rev.* C 96 (2017) 044904



Spectra of emitted particles are very well described by thermal radiation from a collectively moving source

Evidence for thermalization: Yields

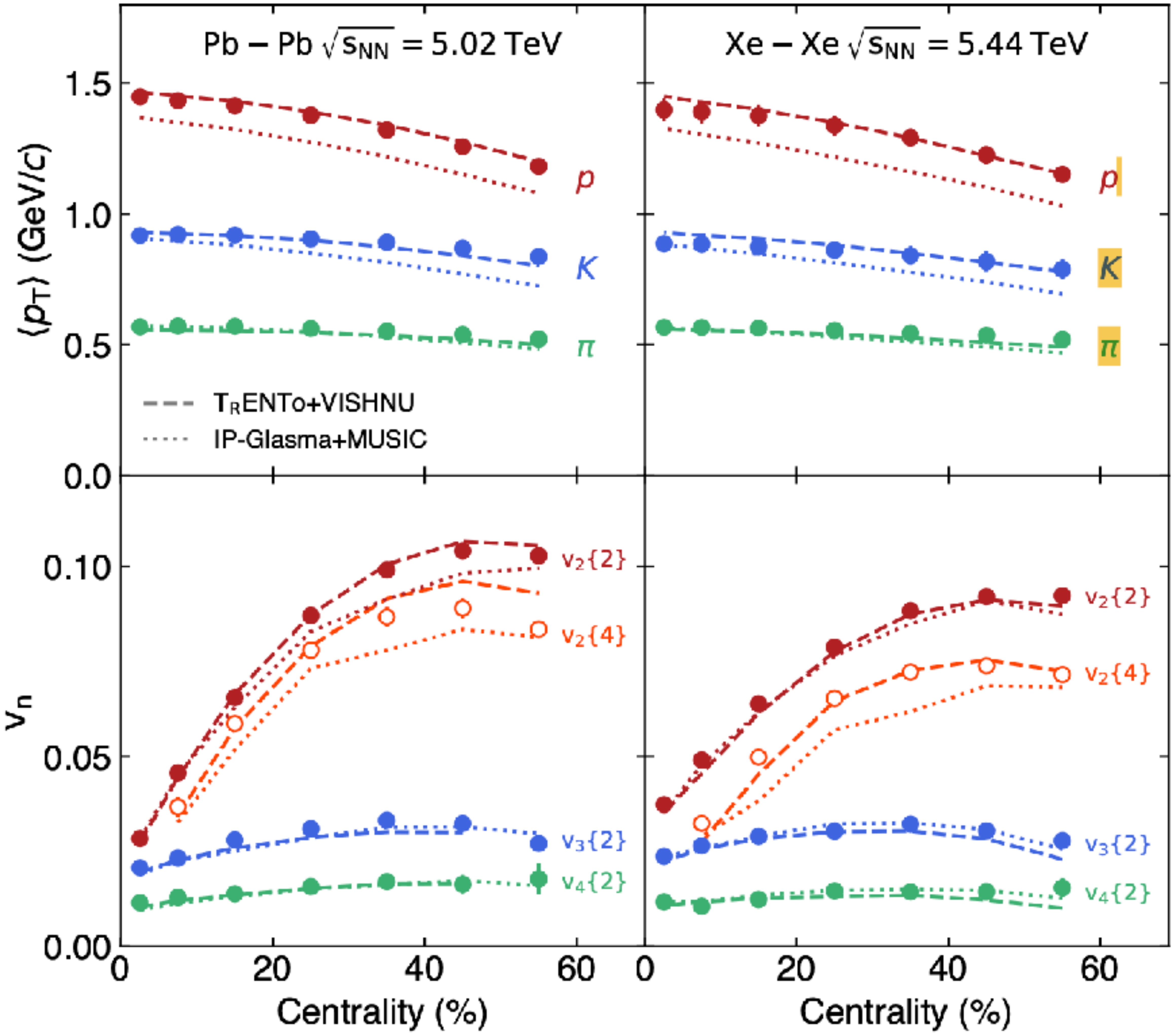
STAR: *Phys. Rev.* C 96 (2017) 044904



Yields of emitted particles are very well described by emission from a common thermal source

Evidence for thermalization: Hydrodynamics

Data: ALICE Collaboration

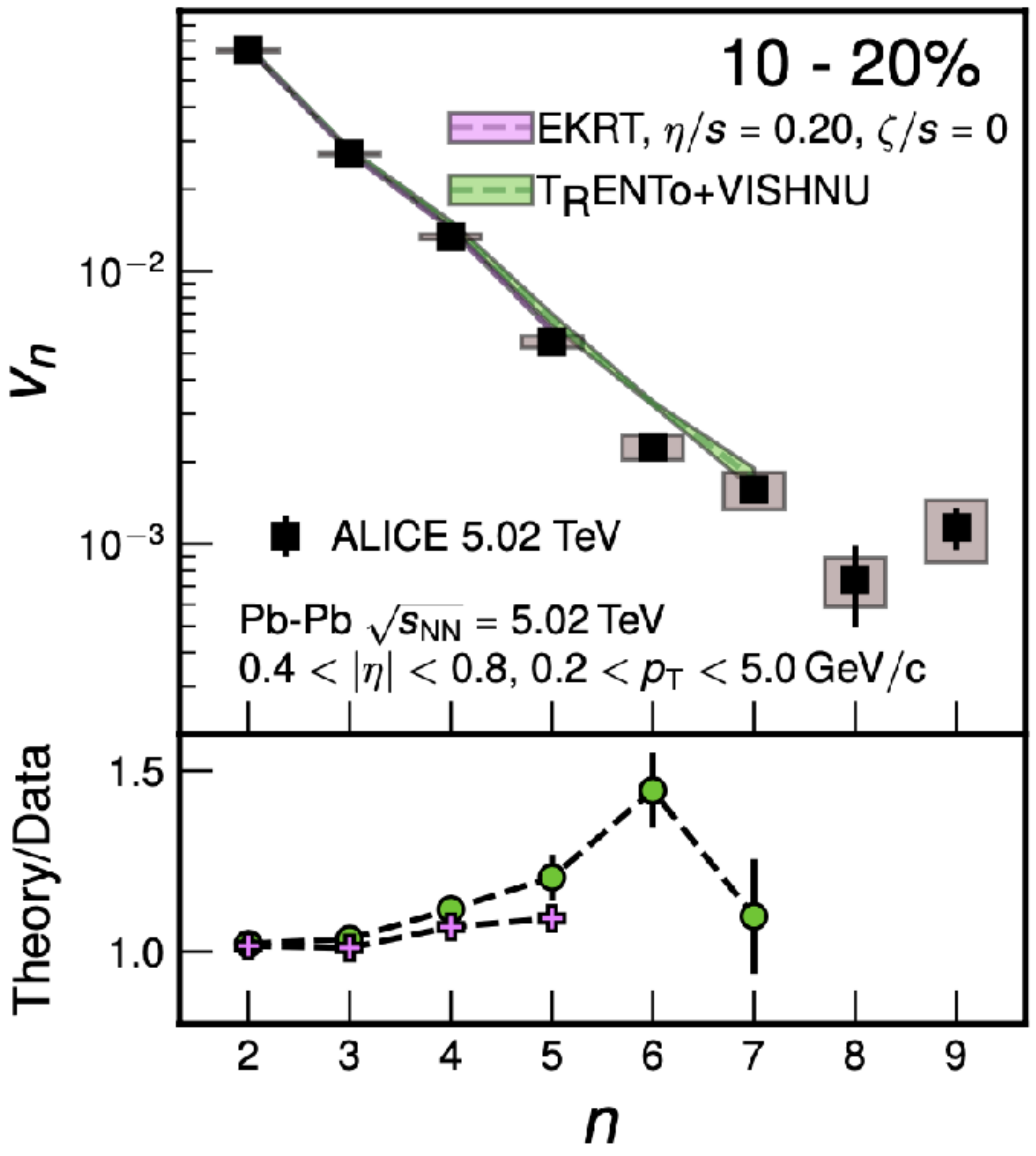


Flow patterns of emitted particles are very well described by relativistic viscous hydrodynamics

$$\frac{dN}{d\phi} \propto 1 + 2\sum_{n=1}^{\infty} v_n \cos\left(n(\phi - \psi_{\mathrm{RP}})\right)$$

Evidence for thermalization: *n*-particle correlations

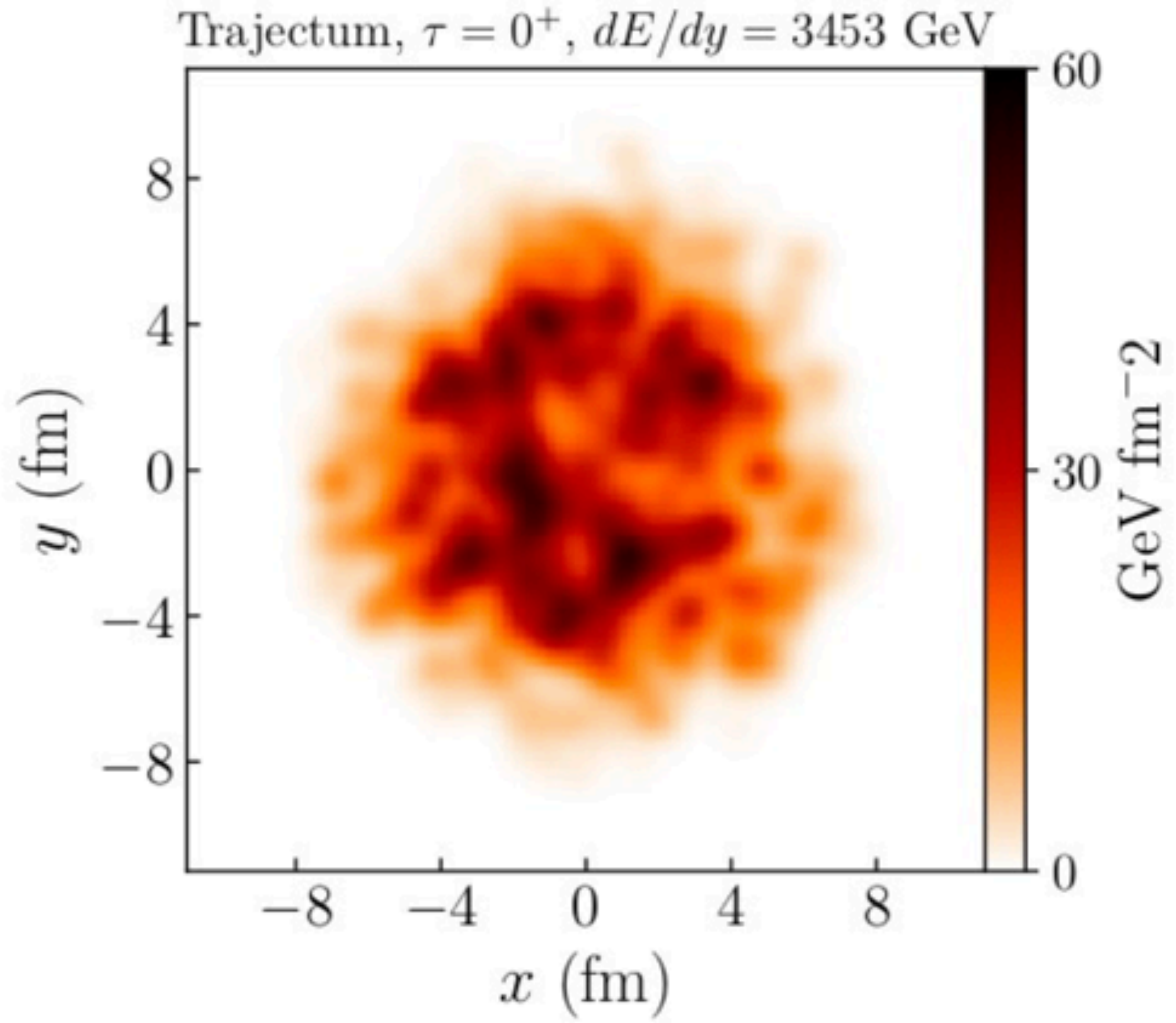
ALICE: *JHEP* 05 (2020) 085



Azimuthal anisotropy “flow” coefficients v_n:

$$\boxed{\frac{dN}{d\phi} \propto 1 + 2\sum_{n=1}^{\infty} v_n \cos\left(n(\phi - \psi_{\mathrm{RP}})\right)}$$

v_n are extracted from n-particle correlations



Sound attenuation during expansion:

$$v_n \propto \varepsilon_n \exp\left(-\frac{4\eta}{3s}\frac{n^2}{TR}\right)$$

Evidence for thermalization

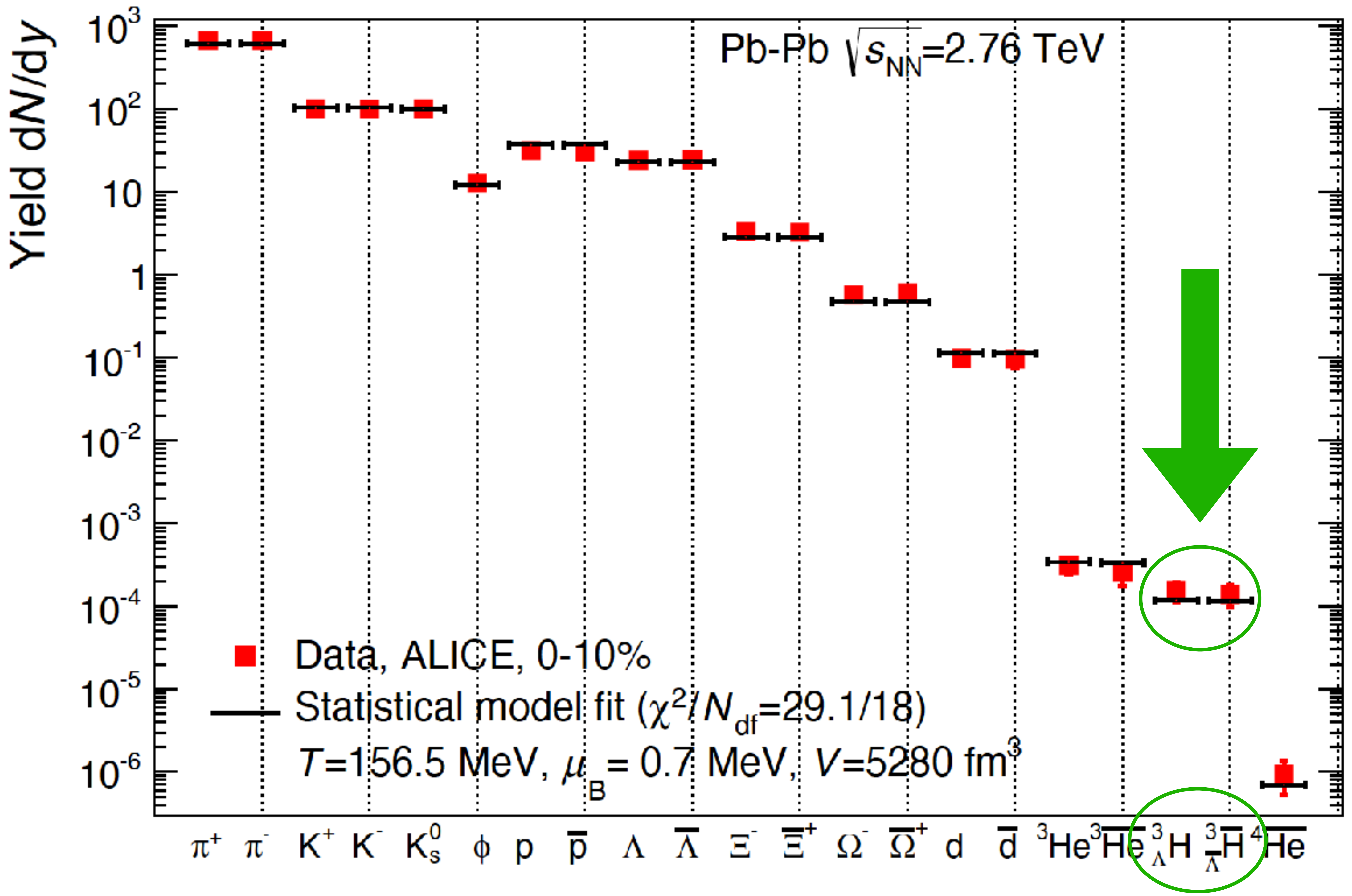
Evidence for the thermalization of matter in a heavy ion collision comes from multiple sources:

- Spectra: Thermal radiation from a collectively moving source.
- Particle yields: Emission from a common thermal system.
- Flow pattern: Relativistic viscous hydrodynamics.
- *n*-particle correlations: Common initial density fluctuations.

Hundreds of studies have failed to find significant deviations from such a picture.

In fact, some have found evidence that seems “too good to be true”.

Thermal particle yield: Is it too good?



Thermal model of particle emission

$$\frac{dN}{dy} \propto \exp\left(-\frac{M_i c^2 - \mu_B A_i}{T}\right)$$

The hyper-triton (pnΛ) is the lightest hyper-nucleus and **very** weakly bound: $B_\Lambda \approx -0.4$ MeV and has an rms radius $R_{rms} > 10$ fm.

How can this particle be emitted with a thermal yield from a fireball of temperature $T = 156$ MeV ?

Does this observation question the validity of the entire thermalization picture, because it is preposterous?

A general principle?

- The unreasonably successful validity of the thermal picture of the final state of a heavy ion collision suggests that general principles are at work, rather than highly specific dynamics.
- There is another known case where such a “miracle” happens: The formation of a black hole.
  - Thermal radiation with Hawking temperature $T_{\mathrm{H}} = \hbar c^3/(8\pi GM)$
  - Bekenstein entropy contained within its horizon $S = 4\pi GM^2/(\hbar c)$ is conjectured to be the largest entropy that can be contained inside this volume.
- If there is a quantum theory of gravity, the process of formation and evaporation of a BH must be described by a unitary S-matrix, the *von Neumann* entropy should remain zero.
- Apparent entropy of Hawking radiation is entanglement entropy.
- The same considerations should apply to relativistic heavy ion collisions.
- Remarkably, over the past decade or so, we have learned that BH thermalization and thermalization in RHI collisions can be mapped into each other by AdS/CFT duality.

AdS/CFT Duality

AdS/CFT duality - holography

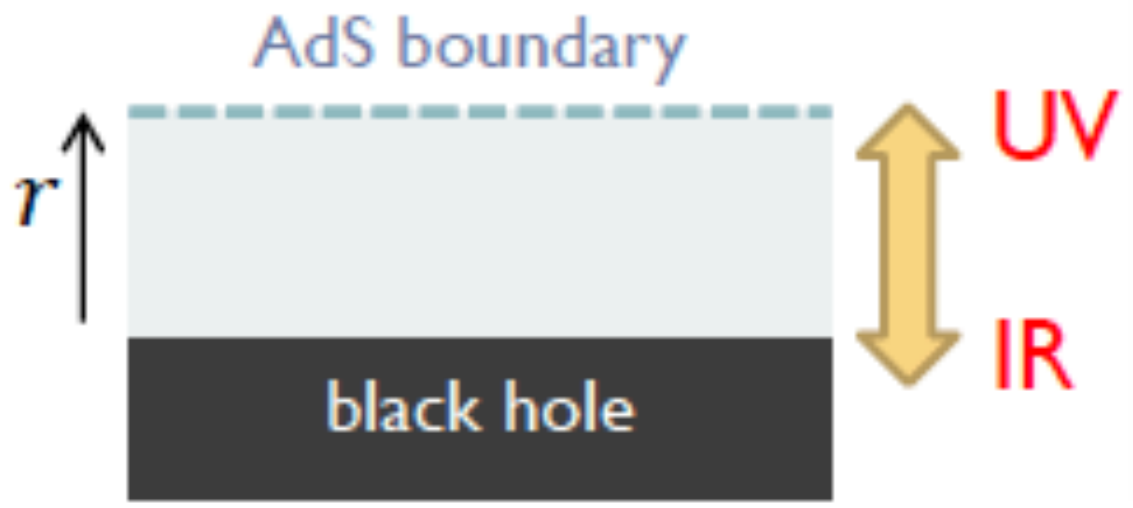
- Naively speaking, rapid equilibration of energy requires strong interactions (in some form) among the constituents of a system. Unfortunately, no rigorous analytical approaches to strongly coupled quantum field theories were known until 1997 (Maldacena).
- A strongly coupled "cousin" of QCD − the SU(N_c) gauge theory with $\mathcal{N}$=4 supersymmetries − is *dual* to a certain weakly coupled string theory on 5-dimensional Anti-de Sitter space (AdS_5).
  - AdS = conformally invariant solution of Einstein's equations with $\Lambda < 0$.
  - In the limit $N_c \to \infty$ the string theory effectively becomes classical gravity, which can be solved using standard analytical and numerical techniques.
- The $\mathcal{N}$=4 SUSY gauge theory "lives" on the boundary of AdS_5 space. The d*uality* means that every observable in the gauge theory can be mapped onto an object in the 5-D gravity theory.
- If AdS_5 space contains a black hole, the boundary gauge theory is thermally excited.

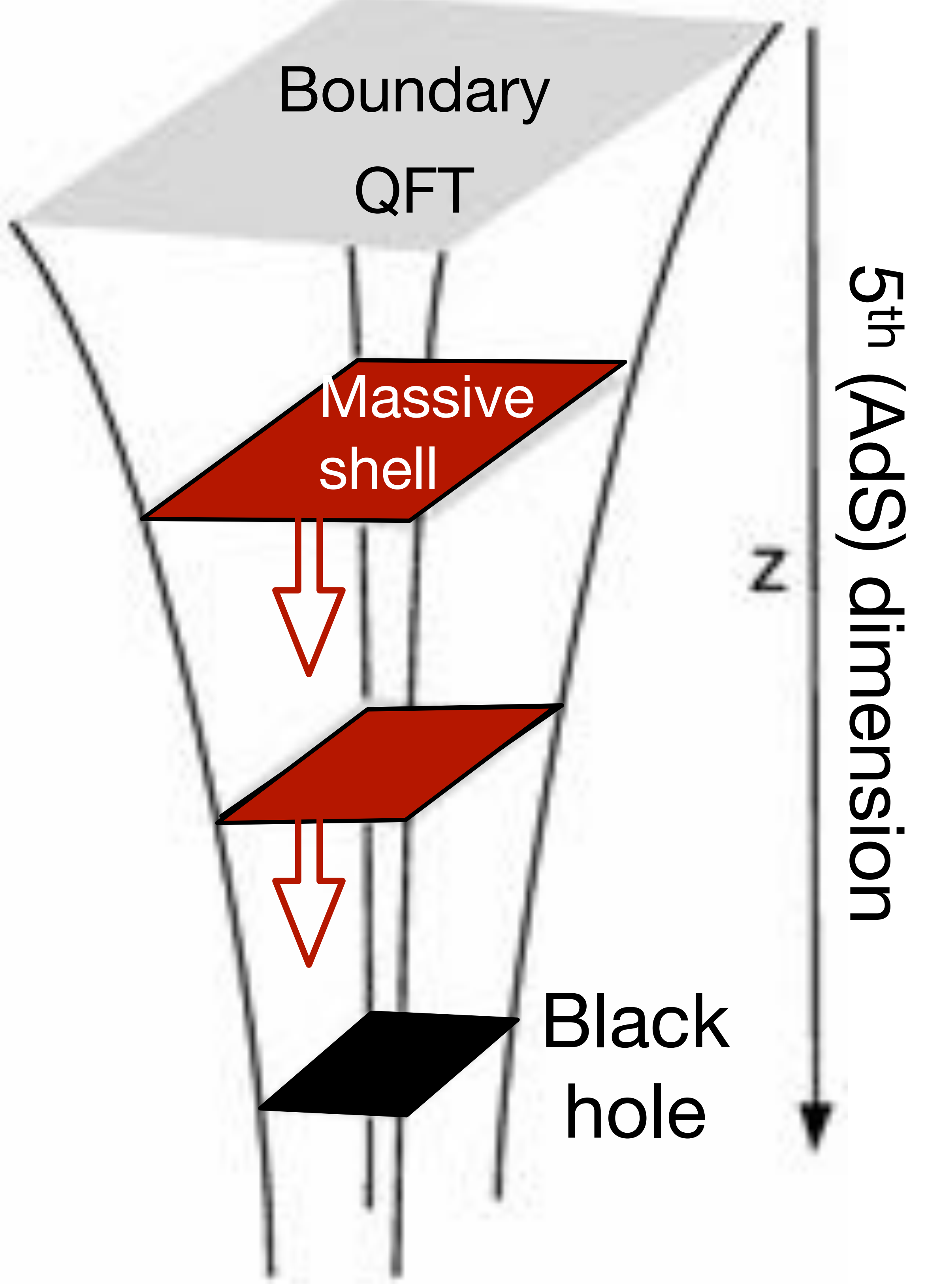
AdS/CFT dictionary

- Want to study strongly coupled phenomena in QCD
- Toy model: $\mathcal{N} = 4\; SU(N_c)$ Super-Yang-Mills (SYM) theory

$\mathcal{N} = 4$ SYM	holo. dual $\Longleftrightarrow$ AdS/CFT	String theory in AdS$_5$
Strong coupling	$\longleftrightarrow$	Weak coupling
Vacuum	$\longleftrightarrow$	Empty AdS$_5$
Thermal state (equilibrated plasma)	$\longleftrightarrow$	AdS BH
Thermalization	$\longleftrightarrow$	BH formation
Heavy ion collision	$\longleftrightarrow$	**Energy injection**

$$\left(\frac{R_{AdS}}{l_s}\right)^4 = g_{YM}^2 N_c \equiv \lambda$$





Strong coupling: $\lambda \gg 1 \quad \rightarrow \quad$ classical gravity

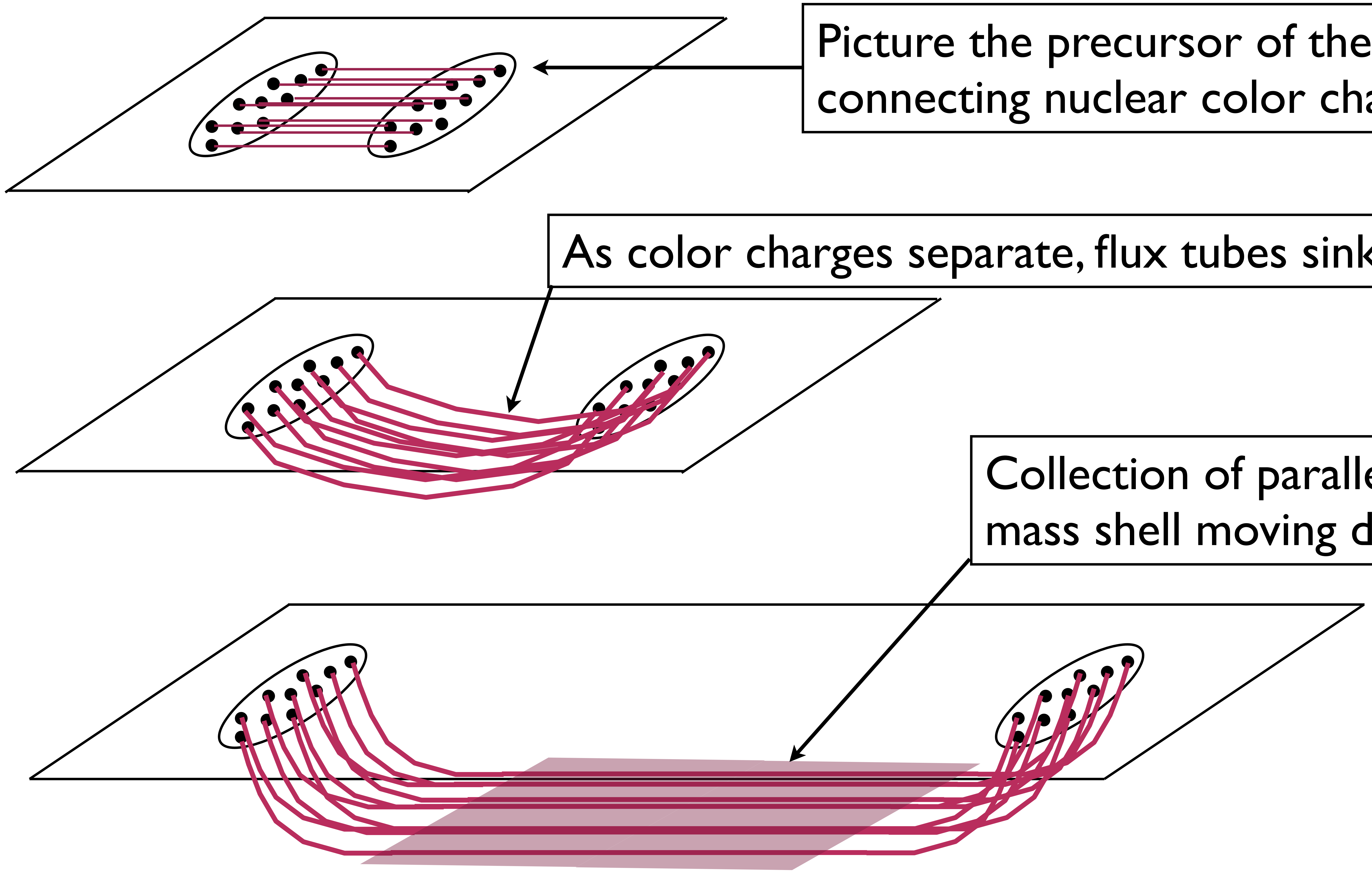
Weak coupling: $g_{\mathrm{YM}} \ll 1 \quad \rightarrow \quad$ Perturb. QCD

Idealized heavy ion collision

Picture the precursor of the QGP as collection of flux tubes connecting nuclear color charges moving apart at high speed.

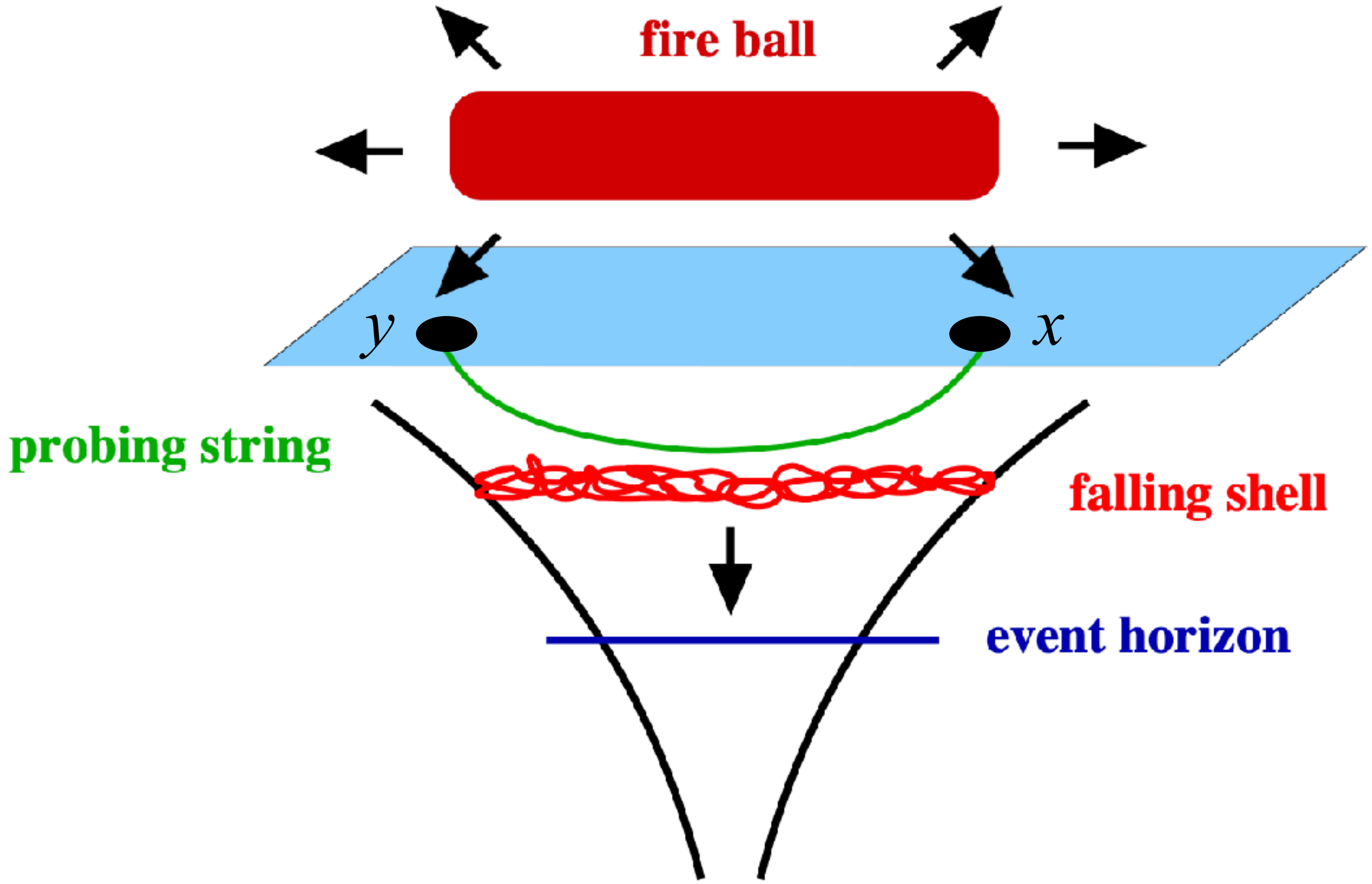
As color charges separate, flux tubes sink into the bulk.

Collection of parallel flux tubes is idealized as thin mass shell moving deeper into the bulk.



When the shell sinks below its Schwarzschild radius, thermalization is reached.

“Holographic” thermalization



The idea:

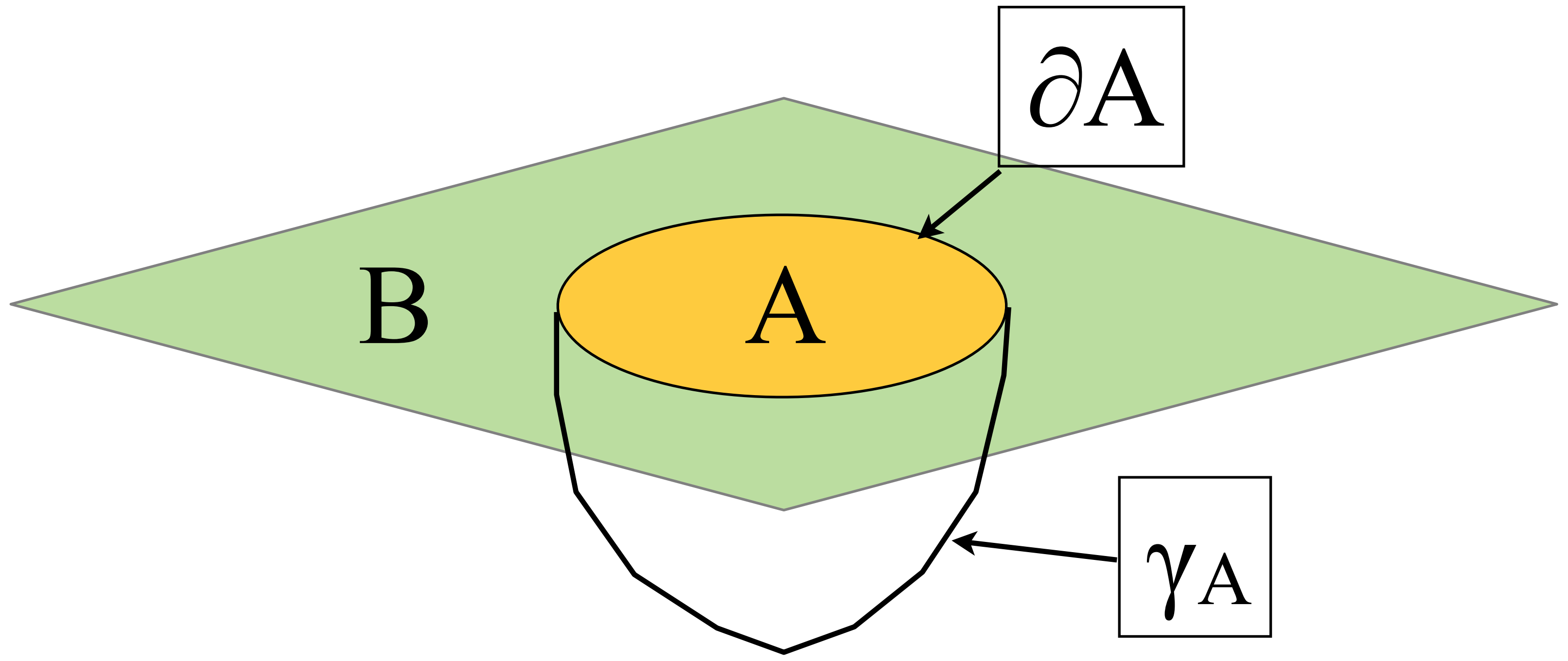
- Probe for thermalization with an observable $\langle O(y)O(x)\rangle$
- Such bi-local observables are needed, e.g., to find the momentum spectrum
- The dual in the AdS_5 geometry is a string that hangs between x and y
- Evaluate in the presence of the falling massive shell

V. Balasubramanian, et al., PRL 106 (2011) 191601; PRD 84 (2011) 026010

Entropy

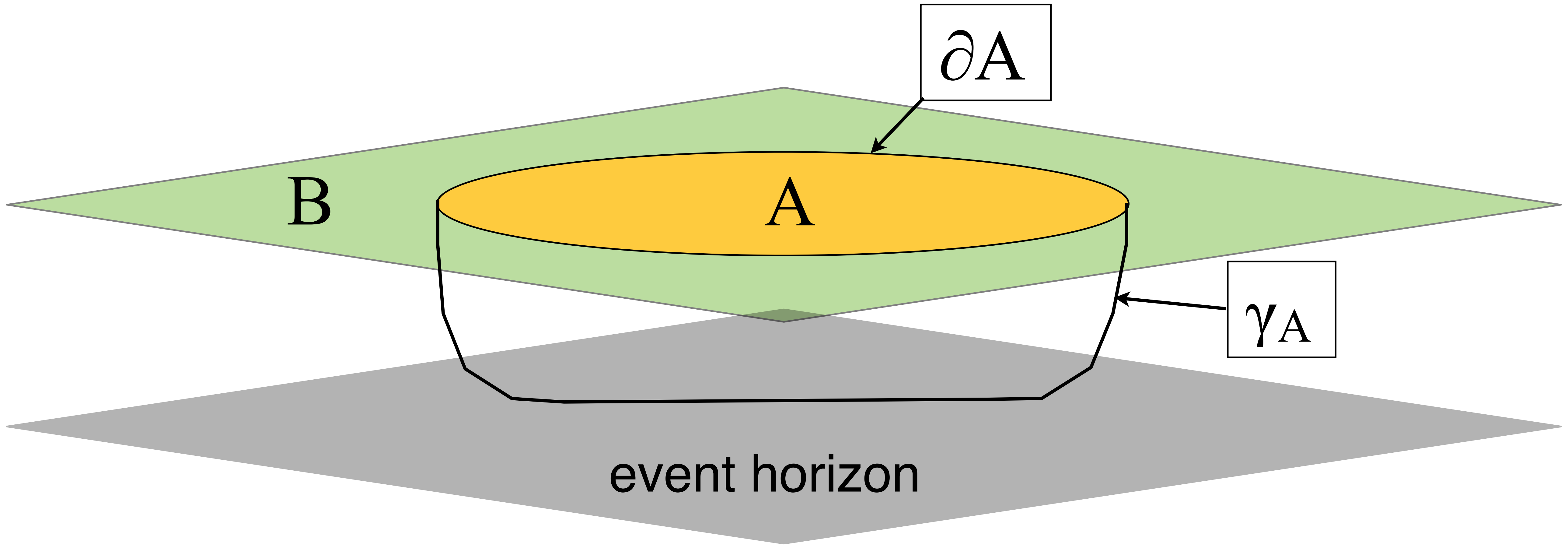
An important quantity is the **entropy S_A** contained in a certain volume A on the boundary.

For a quantum field theory with a holographic gravity dual, S_A can be calculated in the dual theory from the area of the extremal surface γ_A in the bulk that has the same boundary ∂A as A: $\partial(\gamma_A) = \partial A$.



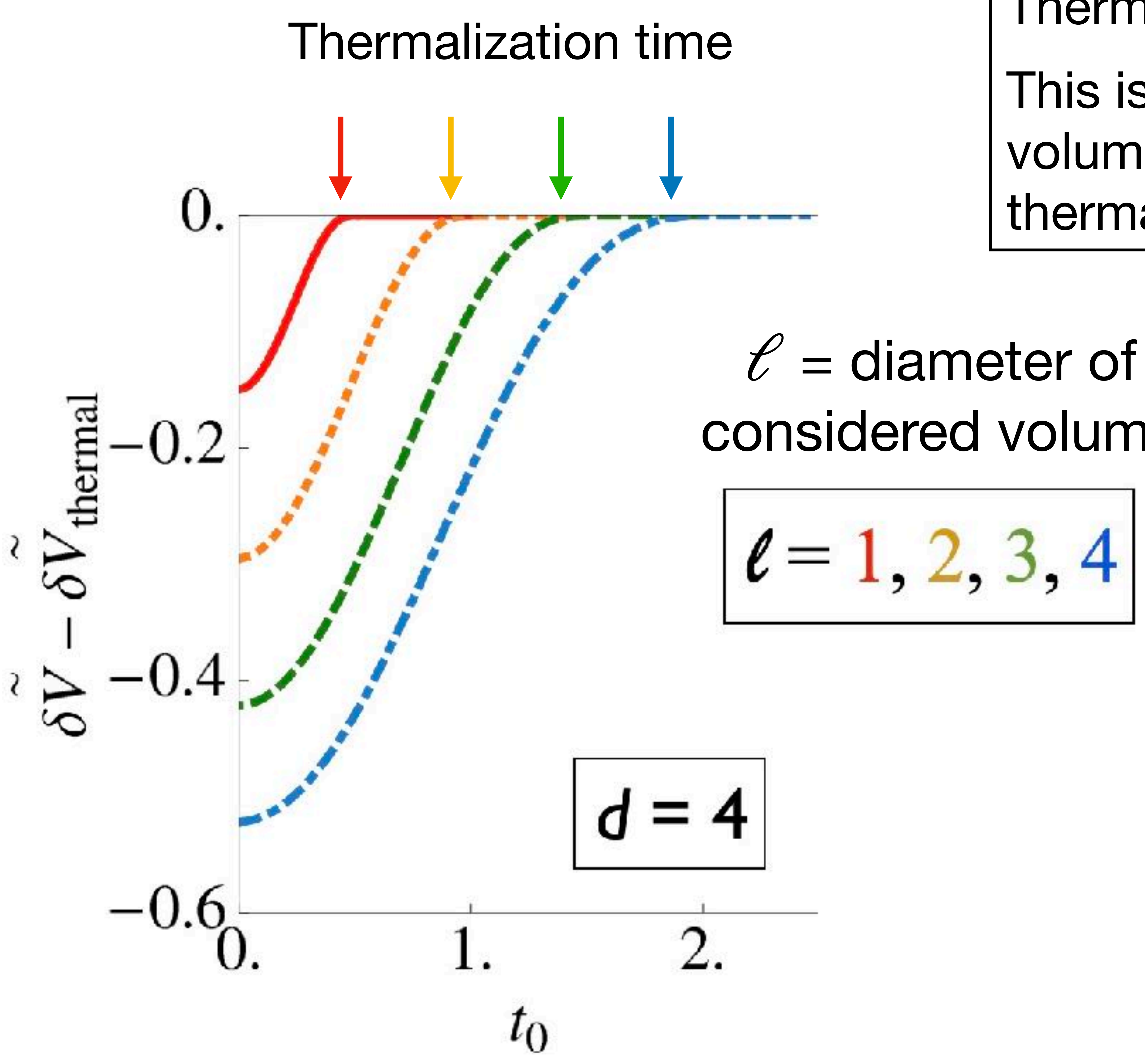
$$S_A = \frac{\text{Area of } \gamma_A}{4G}$$

(Ryu & Takayanagi 2006)



At finite temperature, a BH is present, and the surface γ_A picks up a part of the event horizon, thus accounting for the thermal equilibrium entropy of A.

Entropic thermalization



ℓ = diameter of considered volume

Thermalization time for the (entanglement) entropy is $\tau_{\mathrm{th}} = \ell/2$.

This is time for information to escape from the center of the volume to the surface at the speed of light, I.e. it is the fastest thermalization time compatible with causality.

Rough estimate is $\ell \geq \hbar/T$ and thus:

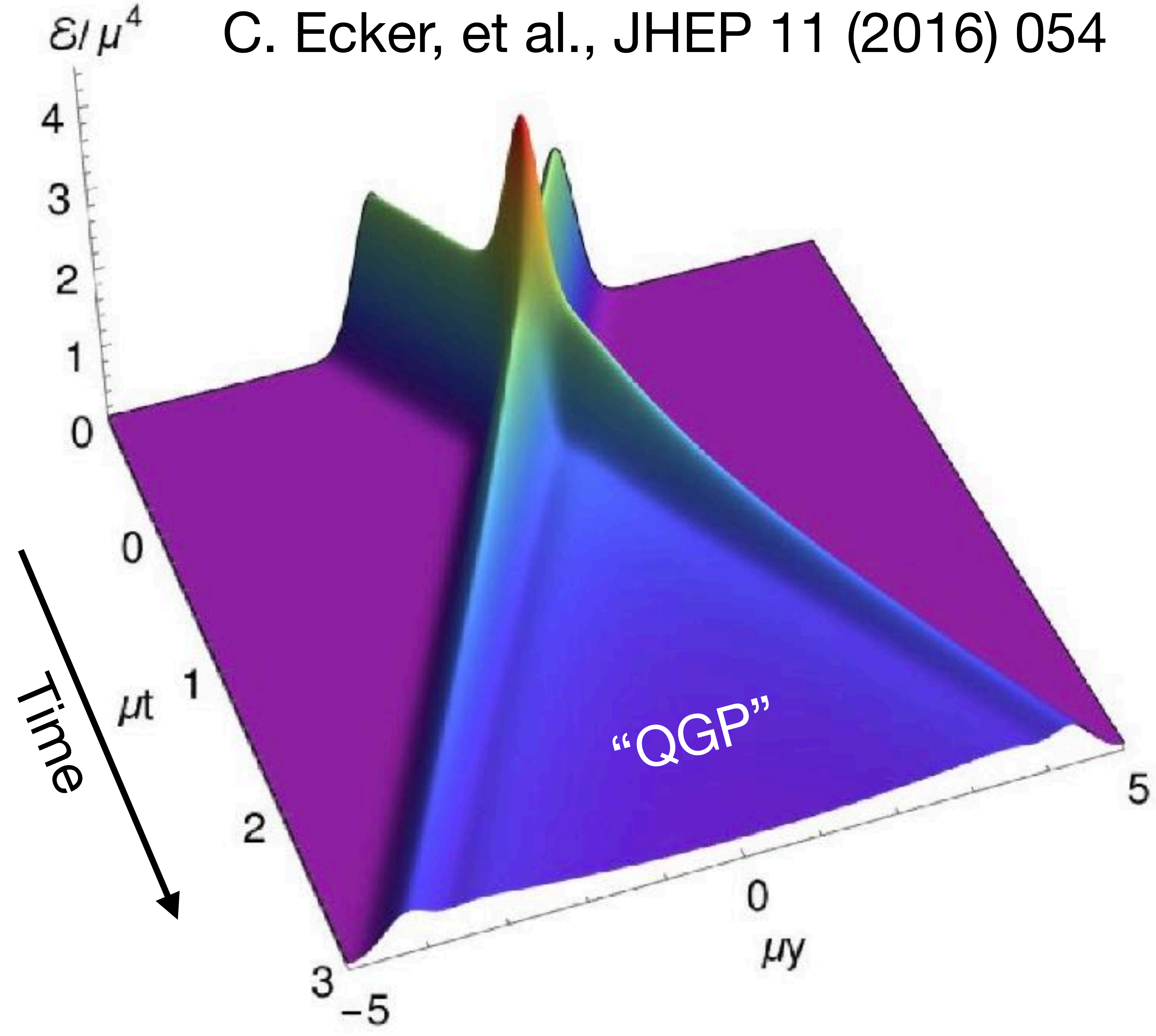
$\tau_{\mathrm{th}} \geq 0.5\ \hbar/T \approx 0.3$ fm/c for $T = 300$ MeV

A heavy ion collision is much more complicated than a sudden quench where unstructured energy is injected into the quantum field. Several groups of theorists have performed increasingly sophisticated numerical calculations where two energetic shock waves collide and studied thermalization via the formation of a black hole horizon in AdS_5.

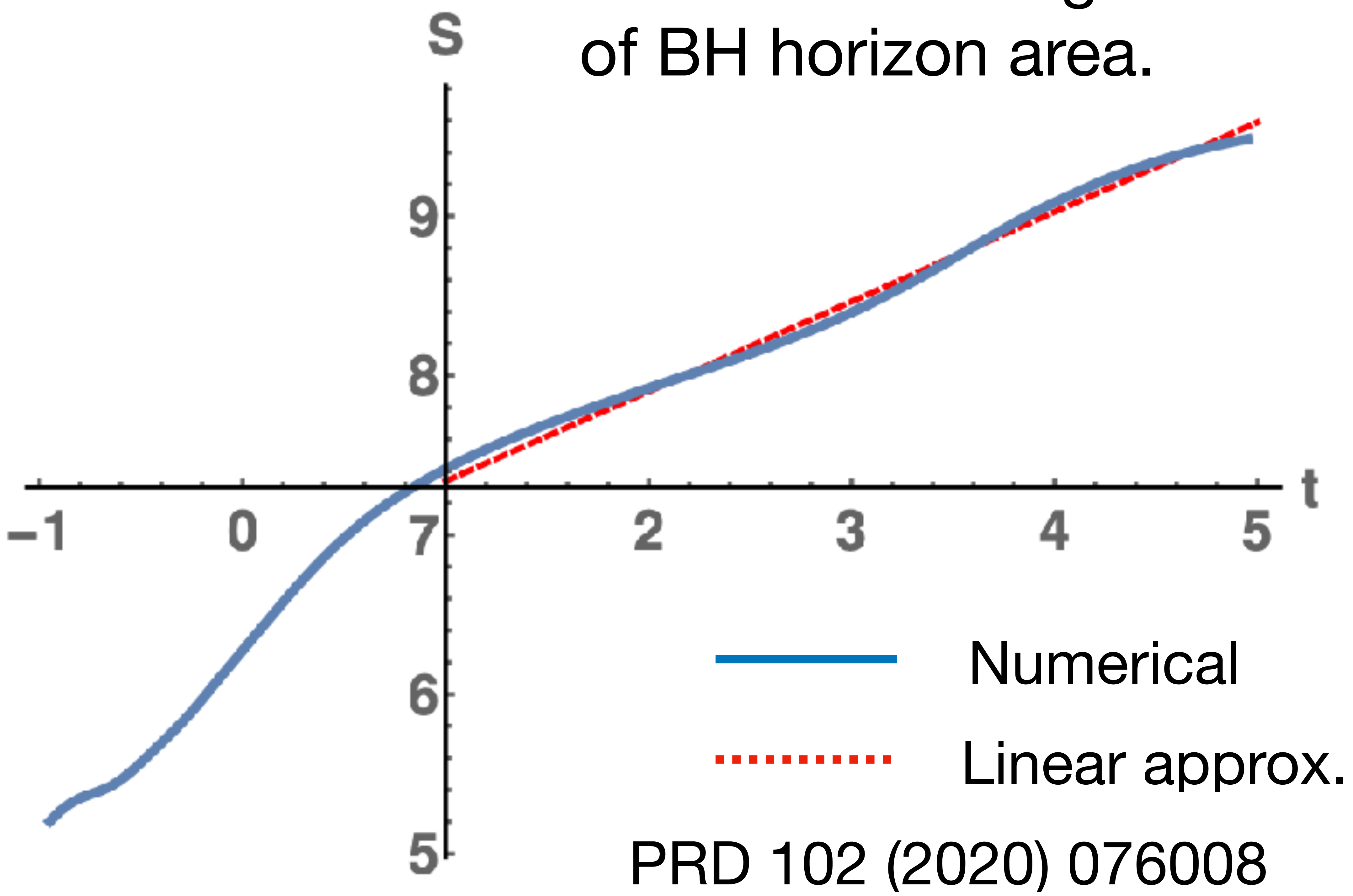
Shock wave collisions

The collision of two energetic narrow shock waves is the “poor person’s” analogue of a relativistic heavy ion collision. The shocks can be varied in amplitude and thickness.

C. Ecker, et al., JHEP 11 (2016) 054



Entropy growth can be calculated from growth of BH horizon area.



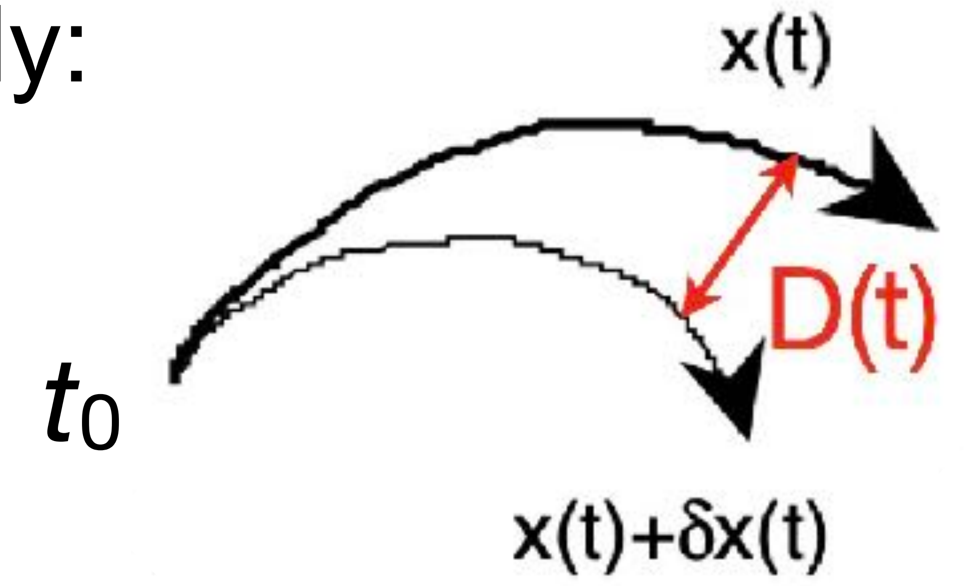
PRD 102 (2020) 076008

Dynamical chaos

Lyapunov exponents - KS entropy

- **A constant growth rate of the observable entropy, i.e. the entropy measured after coarse graining, is a characteristic feature of *chaotic* dynamical systems.**
- Consider two evolutions of such a system starting from slightly different initial conditions $(\vec{x}(t_0), \vec{p}(t_0))$ and $(\vec{x}(t_0) + \delta\vec{x}(t_0), \vec{p}(t_0) + \delta\vec{p}(t_0))$. A dynamical system is *chaotic* if the distance in phase space between the two systems grows exponentially:

$$D(t) = \sqrt{|\delta\vec{x}(t)|^2 + |\delta\vec{p}(t)|^2} = D_0 e^{\lambda t}$$



- λ is called the (largest) *Lyapunov exponent*.
- More generally, one can construct a spectrum of modes around the original trajectory in phase space and obtain the associated spectrum of Lyapunov exponents λ_i. The rate of growth of the coarse grained entropy is known as the Kolmogorov-Sinai (KS) entropy h_{KS}. It is given by

$$dS/dt = h_{\mathrm{KS}} \equiv \sum_{\lambda_i > 0} \lambda_i .$$

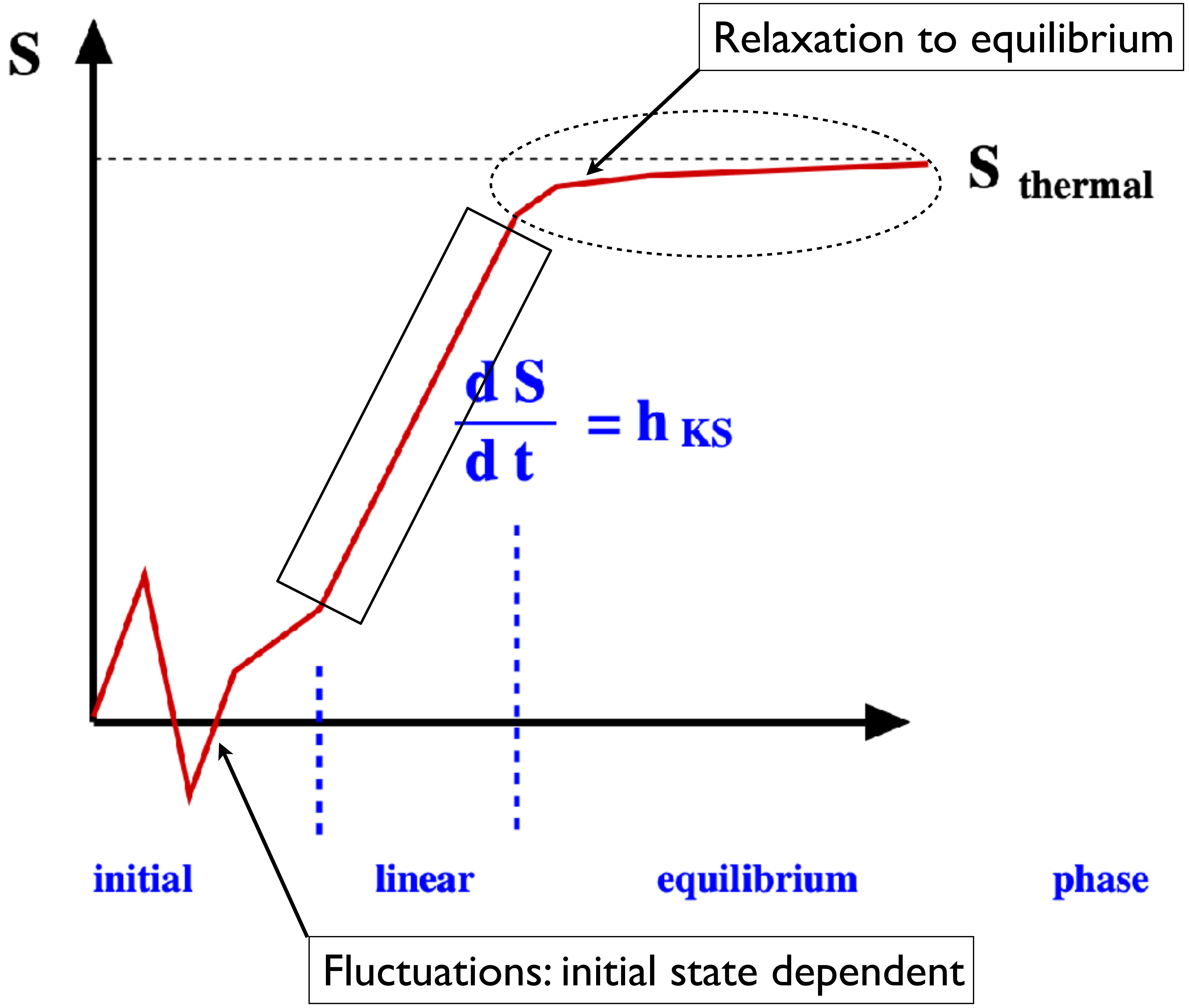
Thermalization of a chaotic system

Depending on the size of initial fluctuations, after some initial period, the measurable entropy of the system grows linearly with time:

$$dS/dt = h_{\mathrm{KS}} \equiv \sum_{\lambda_i > 0} \lambda_i.$$

After a time $\tau_{\mathrm{eq}} = S_{\mathrm{eq}}/h_{\mathrm{KS}}$, the entropy of the system approaches the value of the entropy in thermal equilibrium, and further growth is impossible because the volume of accessible phase space at fixed total energy is finite.

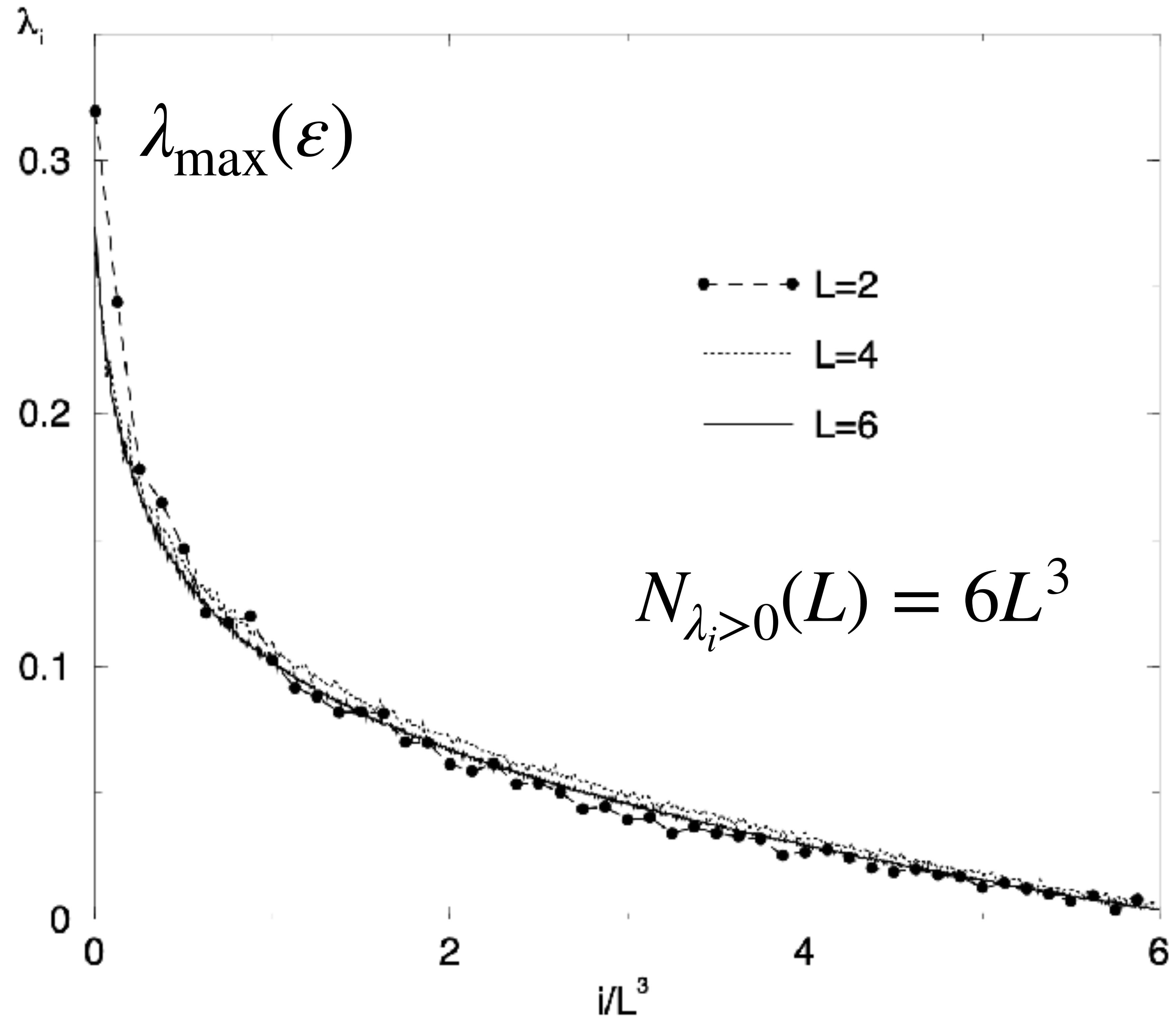
This behavior can be calculated numerically in the classical limit of field theory.



QCD is chaotic

- The nonlinear Yang Mills theory SU(3), which underpins QCD, has been shown to be extensively chaotic.
- All modes in Yang-Mills phase space (A_i^a, E_i^a), except those corresponding to conservation laws, have nonzero Lyapunov exponents. In particular, plane wave solutions are exponentially unstable against perturbations.
- The KS entropy grows linearly with volume, i.e. there is a finite growth rate of the entropy density, and increases with energy density:

$$h_{\mathrm{KS}}(\varepsilon) \sim L^3 \lambda_{\max}(\varepsilon)$$



- Detailed study for SU(2) in: PRD 82 (2010) 114015

How chaotic is QCD?

- Maldacena, Shenker, and Stanford [JHEP 08 (2016) 106] argued that there is an upper bound on Lyapunov exponents: $\lambda \leq 2\pi T$ (*T* is the equilibrium temperature)
- Our numerical simulations for the SU(3) gauge theory found [PRD 52 (1995) 1260]

$$\lambda_{\max} = 0.53 g^2 T \approx 2T$$

  for a typical gauge coupling constant $\alpha_s = g^2/4\pi = 0.3$.

- Thermalization in QCD at realistic coupling may thus be about three times slower than at infinitely strong coupling as realized in BH formation or AdS/CFT, but still gives a rather short thermalization time ≈ 1 fm/*c* for the entropy of the QGP.
- One *caveat*: The influence of quarks on QCD chaos has not been studied, but there are many indications that the presence of quarks does not suppress it.

Husimi coarse graining

A minimal coarse-graining of a quantum system is achieved by projecting the density matrix onto a coherent state (Husimi ["*Fushimi*"] 1940). This procedure smears the Wigner function of the system with a minimum-uncertainty Gaussian in phase space:

$$H_\Delta(p,x;t) \equiv \int \frac{dp'\,dx'}{\pi\hbar} \exp\left(-\frac{1}{\hbar\Delta}(p-p')^2 - \frac{\Delta}{\hbar}(x-x')^2\right) W(p',x';t)$$

The smearing encodes the uncertainty introduced by a generic measurement of x and p.

$H_\Delta(p,x) \geq 0$ and can thus be used to define a coarse grained entropy (Wehrl, 1978):

$$S_{\mathrm{H},\Delta}(t) = -\int \frac{dp\,dx}{2\pi\hbar} H_\Delta(p,x;t) \ln H_\Delta(p,x;t)$$

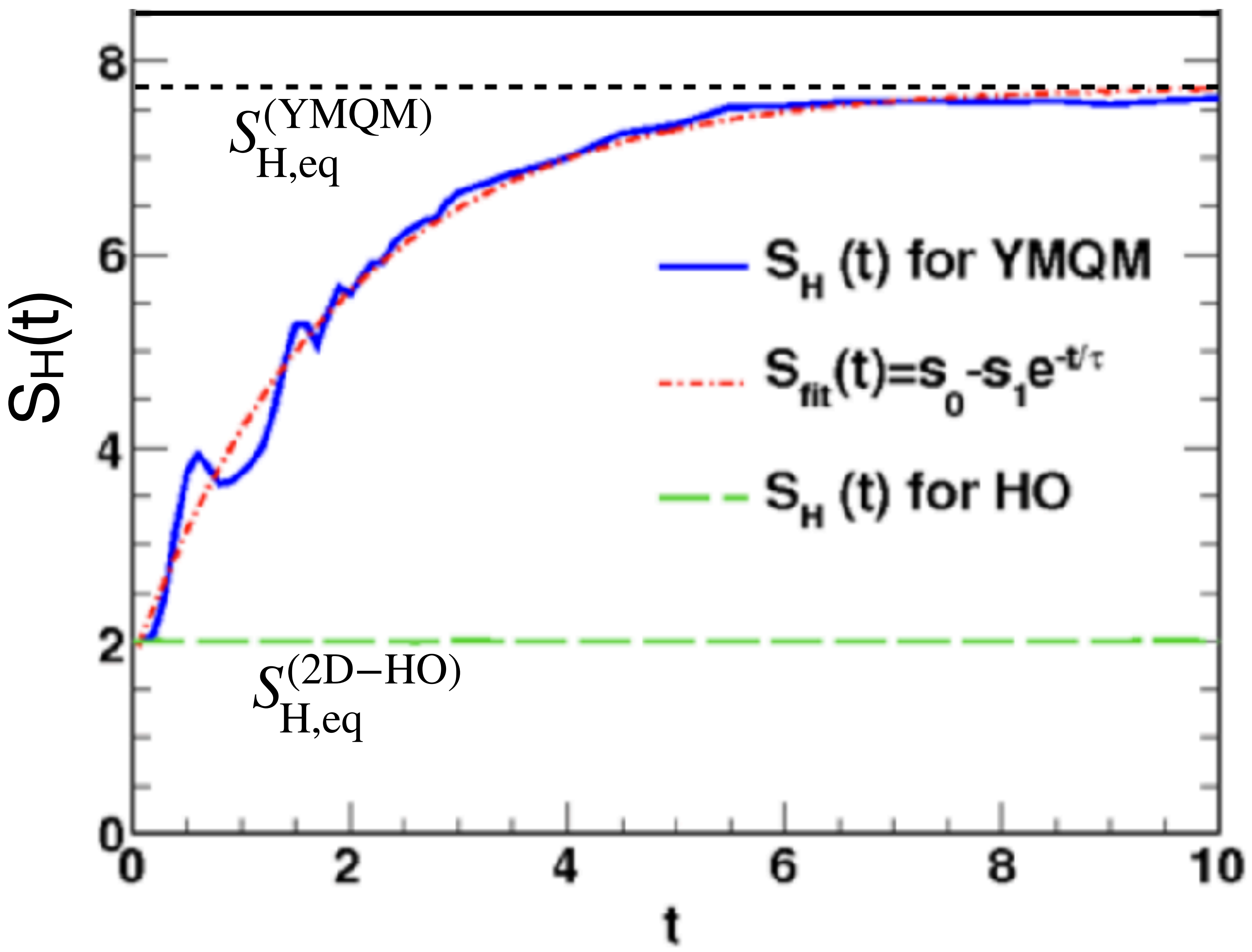
It has the property:

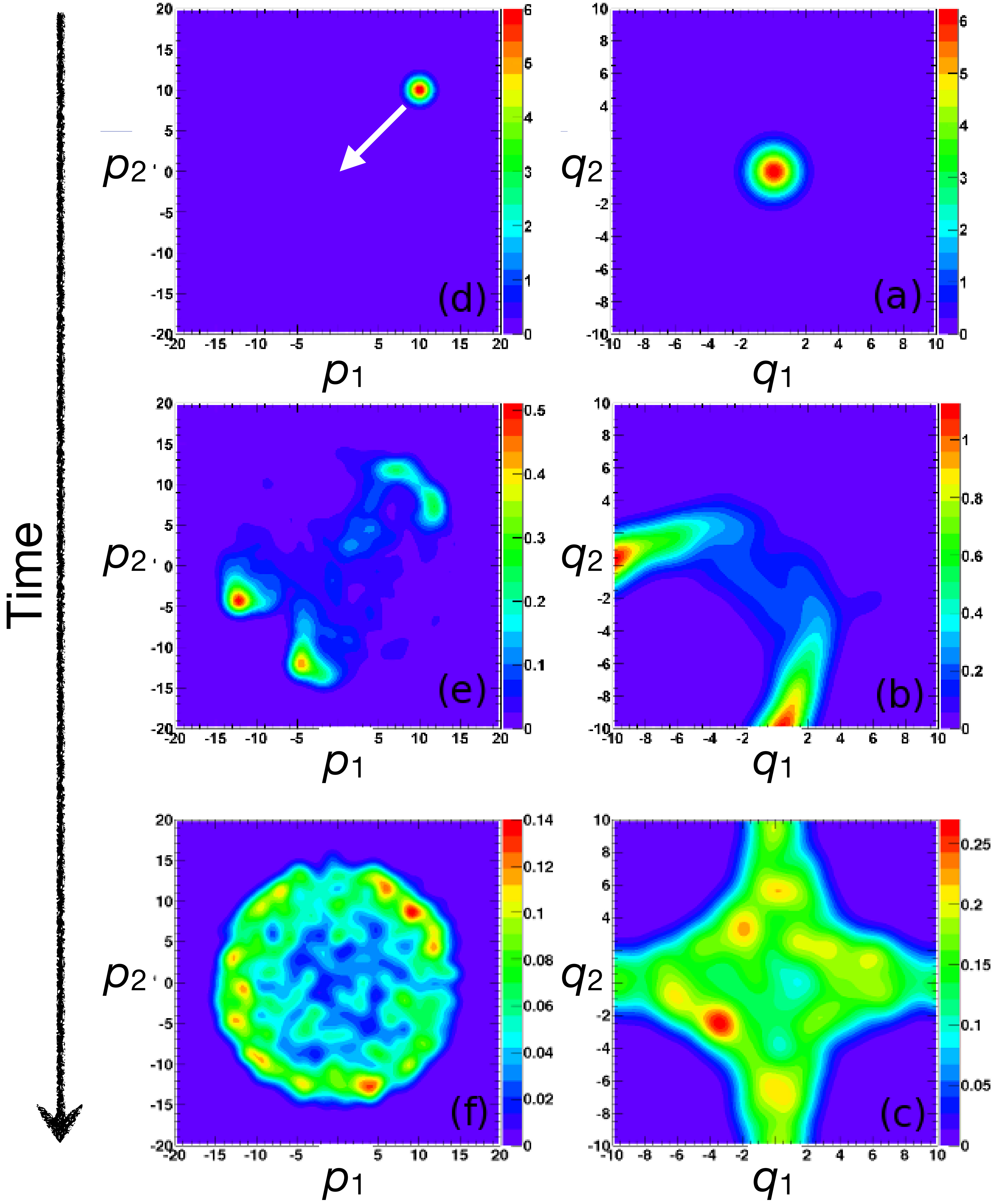
$$\frac{\partial S_{H,\Delta}}{\partial t} = h_{\mathrm{KS}} = \sum_{\lambda_i > 0} \lambda_i$$

YMQM model

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}g^2 q_1^2 q_2^2$$

Arises as infrared limit of SU(2) gauge theory.

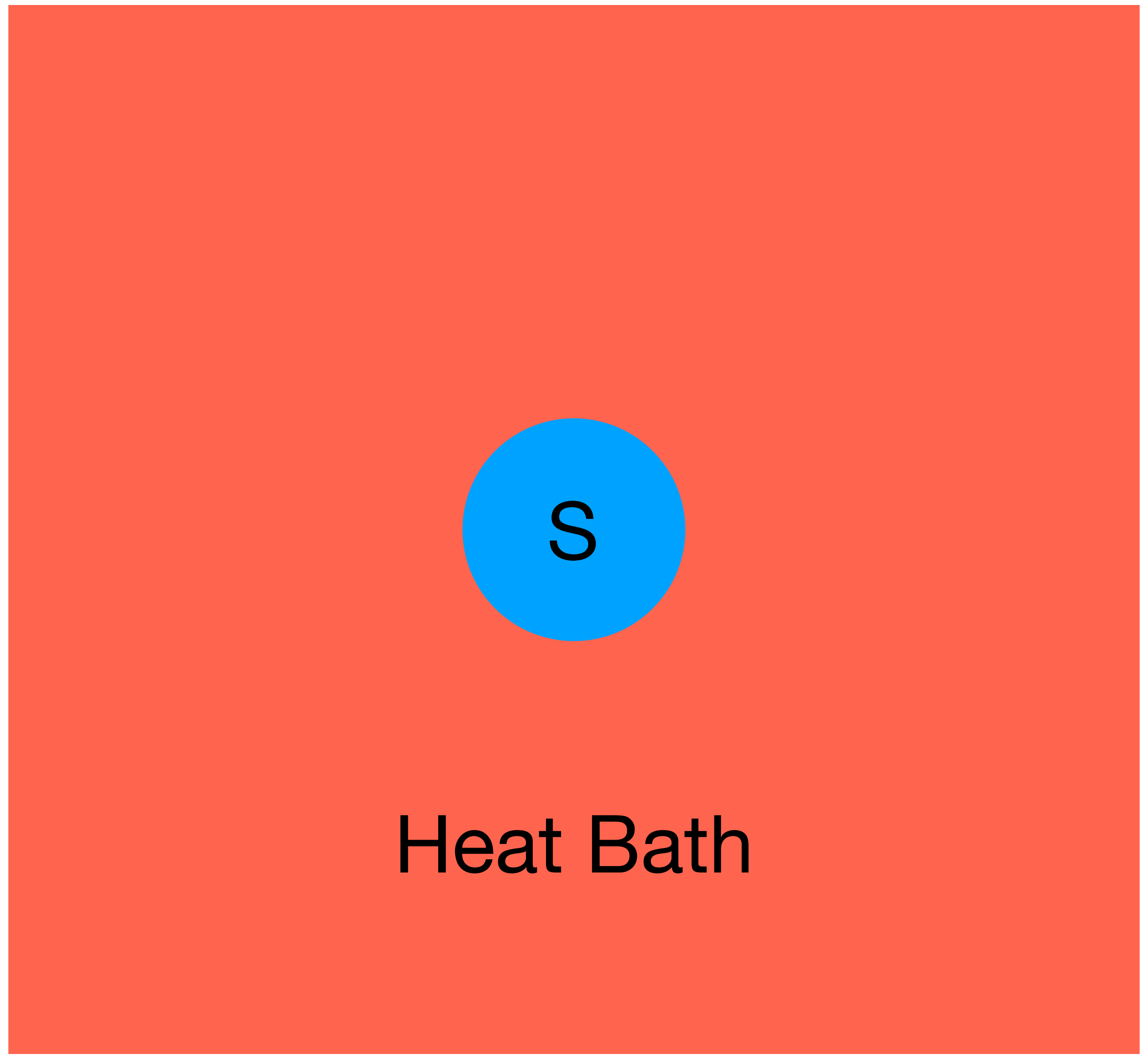




Is *Thermal Equilibrium* just a Mirage?

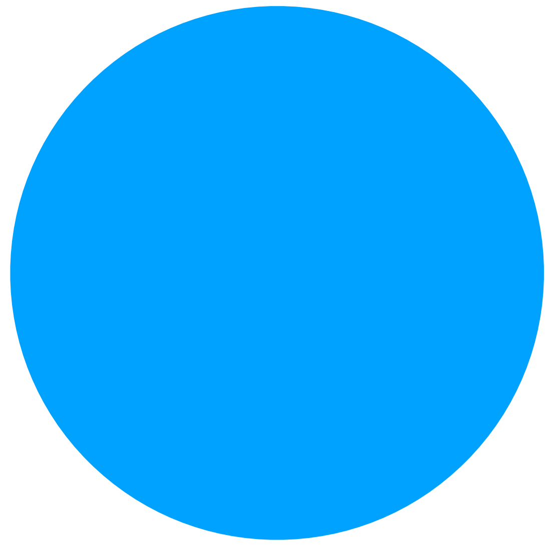
Heat bath vs. isolated systems

System in contact with a heat bath



Quantum state of System (S) becomes entangled with quantum state of the Bath (B), and the entanglement entropy $S = -\,\mathrm{Tr}\,(\rho_S \ln \rho_S)$ with $\rho_S = \mathrm{Tr}_B(\rho)$ approaches the thermal entropy.

Isolated system



What does it mean to say that an isolated system becomes “thermal”?

Where is the information loss that corresponds to entropy?

In classical physics it arises from the interplay between chaotic dynamics and coarse graining.

What is the analogue in the world of quantum physics?

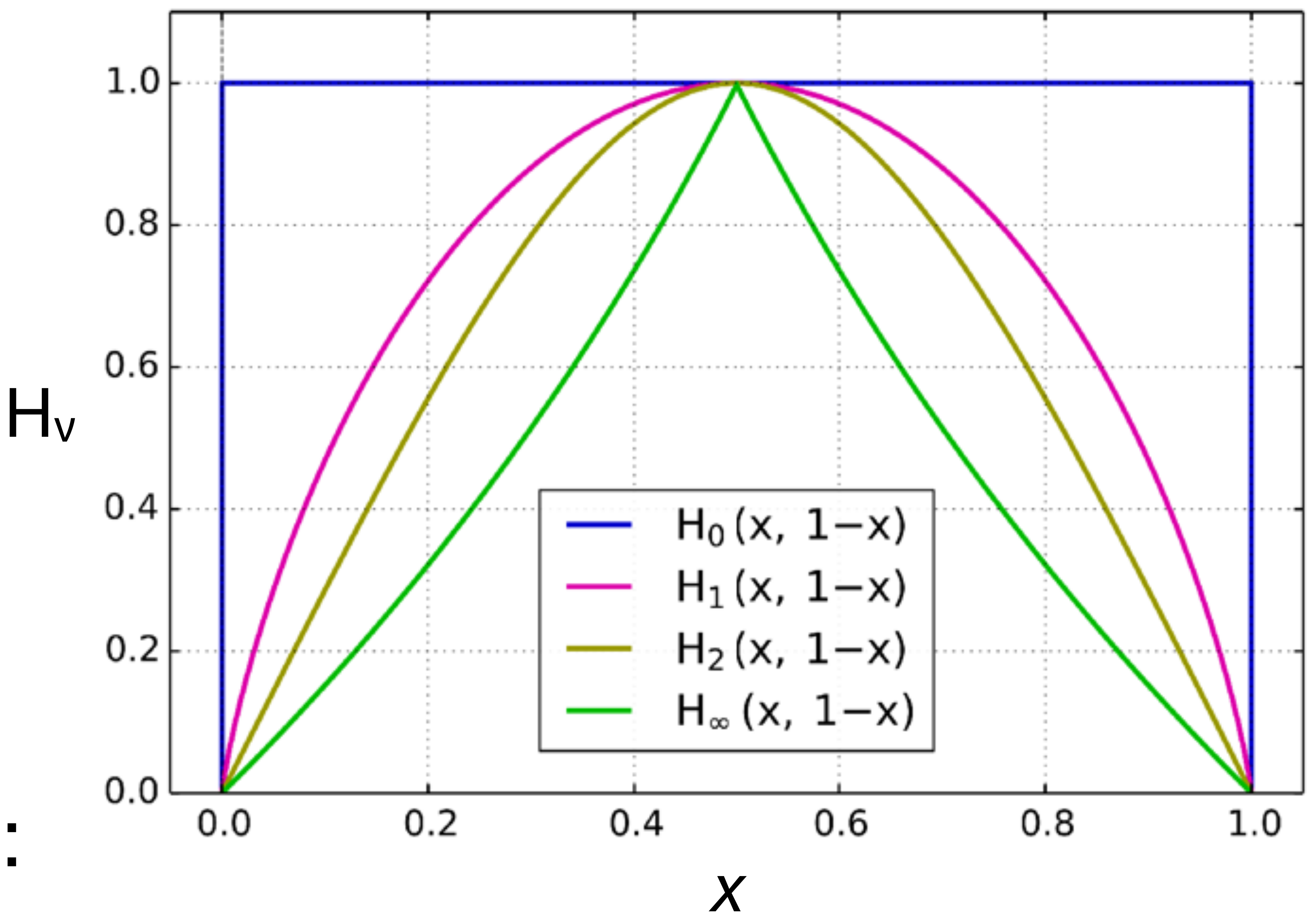
One viewpoint is that an isolated quantum system never **is** thermal, but it can **appear** thermal for all practical purposes. If the system starts out as a pure quantum state, it will always remain a pure quantum state, but measured aspects of it may appear thermal because of entanglement with the unmeasured parts of the system.

Quick primer on entropy

- A general (mixed) state of a quantum system is characterized by a density matrix ρ with $\mathrm{Tr}(\rho) = 1$.
- The *Rényi entropy* or order ν is defined as

$$H_\nu = -\frac{1}{1-\nu}\log(\mathrm{Tr}\,\rho^\nu)$$

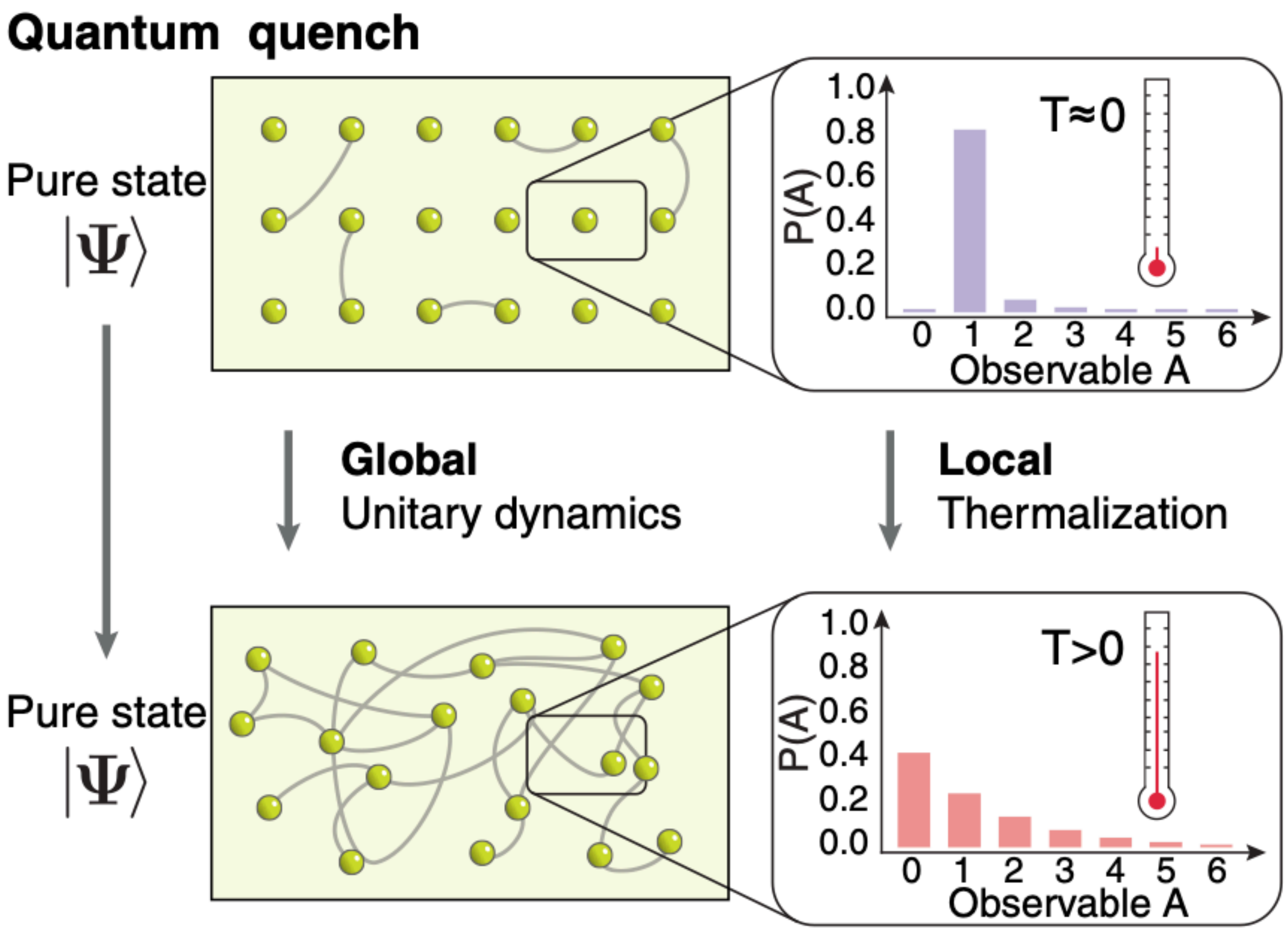
- For $\nu = 1$, this becomes the usual von Neumann (Shannon) entropy: $S = -\,\mathrm{Tr}\,(\rho\log\rho)$.
- The case $\nu = 2$ is often simply called Rényi entropy: $S_{\mathrm{R}} = -\log(\mathrm{Tr}\,\rho^2)$.
- For a pure state: $\rho = |\Psi\rangle\langle\Psi|$ one has $\rho = \rho^2 = \cdots$ and thus $S = S_{\mathrm{R}} = 0$.
- One often says that S_{R} measures the *purity* of the quantum state.



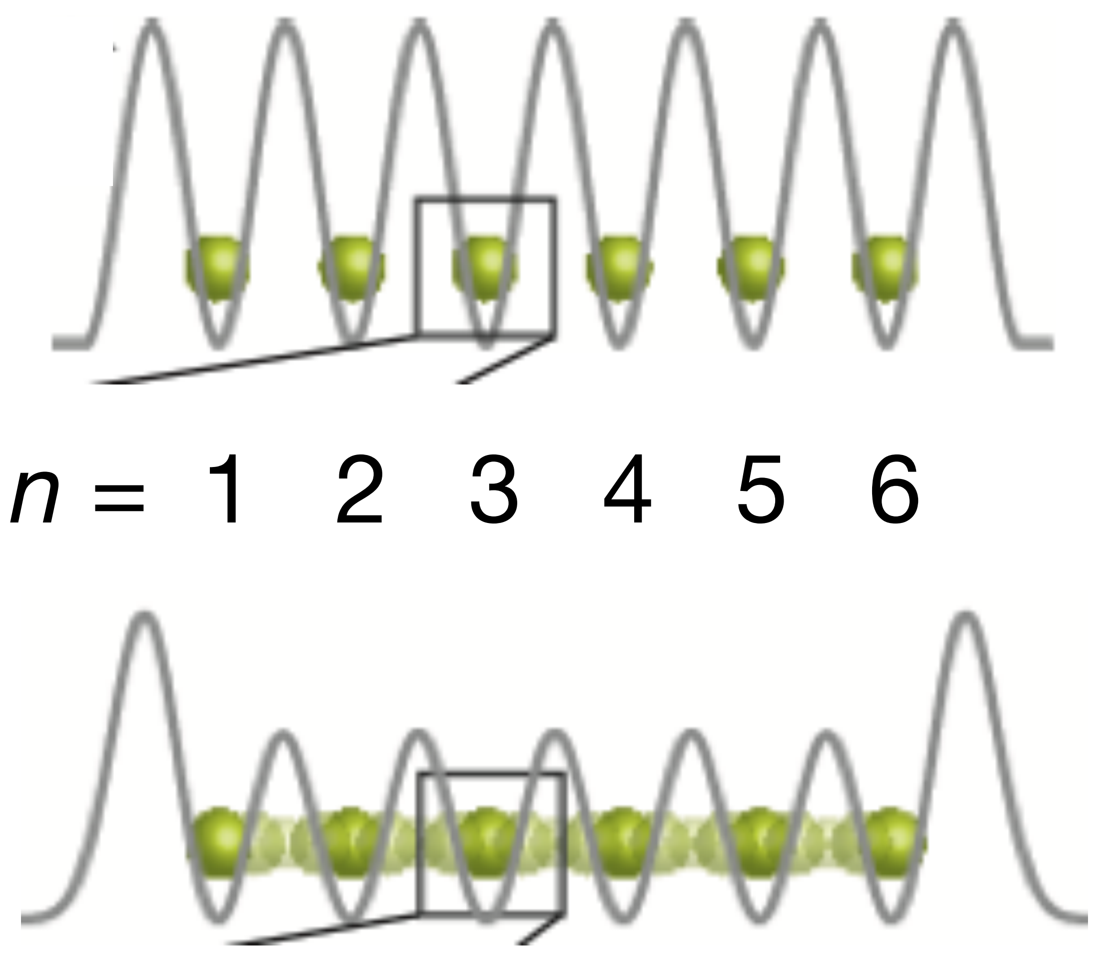
Generalized *Rényi entropy* for a binary random variable with likelihood of outcome $P = (x, 1\text{-}x)$ as function of x. H_1 is the von Neumann entropy; H_2 is the common *Rényi* entropy.

Kaufman *et al.* experiment

A.M. Kaufman, *et al.*, *Science* 353 (2016) 794



Idea: Isolate six ^{87}Rb atoms in a Bose condensate and observe their dynamics under a sudden change of the Hamiltonian (quantum quench).



Before quench:

Each atom is confined to its specific site.

After quench:

Atoms can easily tunnel between sites.

Measure occupation number for each site n.

Approach to thermality

How close a subsystem is to being thermal can be measured in various ways.

One measure is the "trace distance":

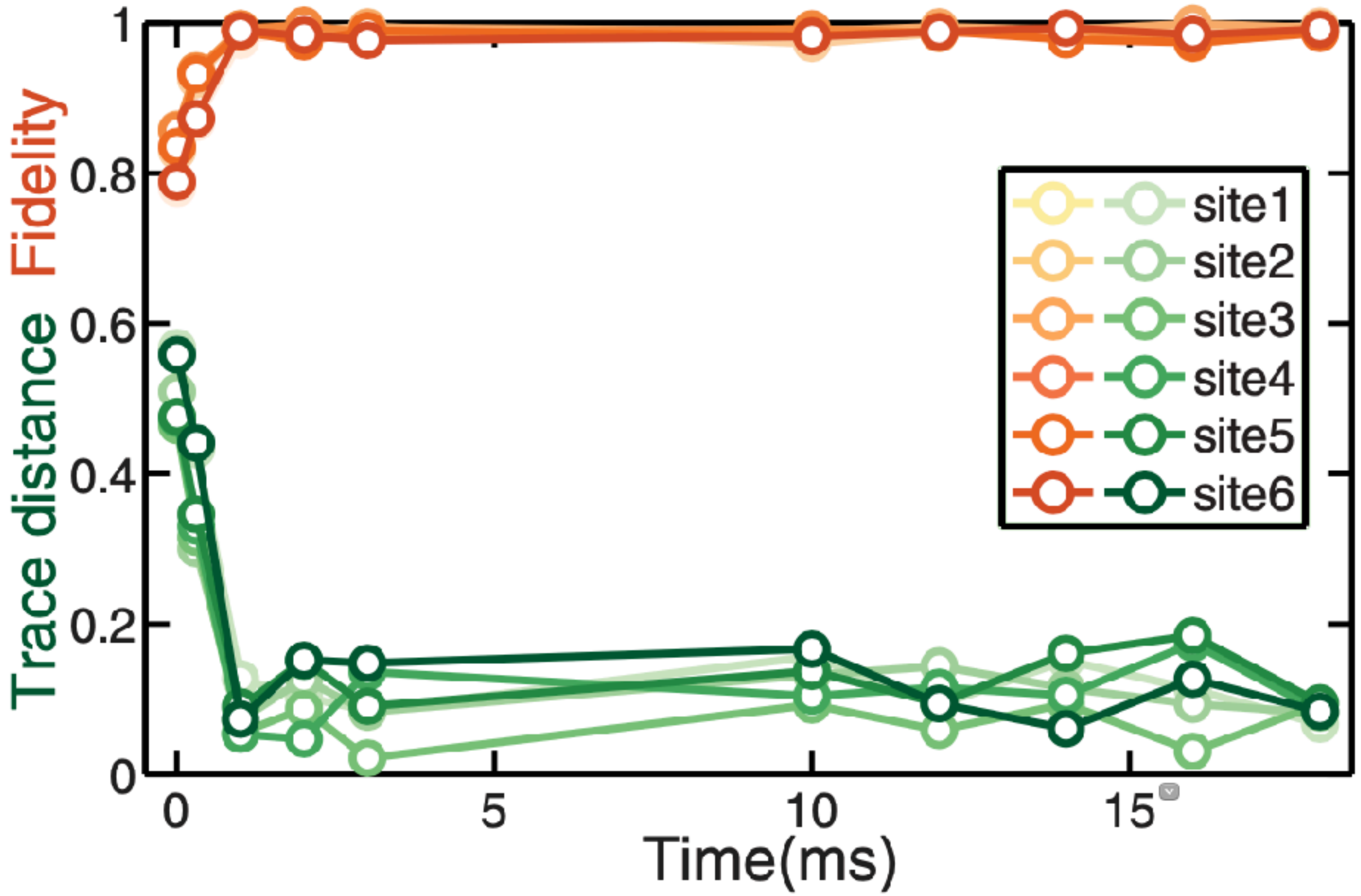
$$D_{\rm tr} = \frac{1}{2}{\rm Tr}\left(|\rho_A^{(T)} - \rho_A|\right)$$

Another measure is the "quantum fidelity":

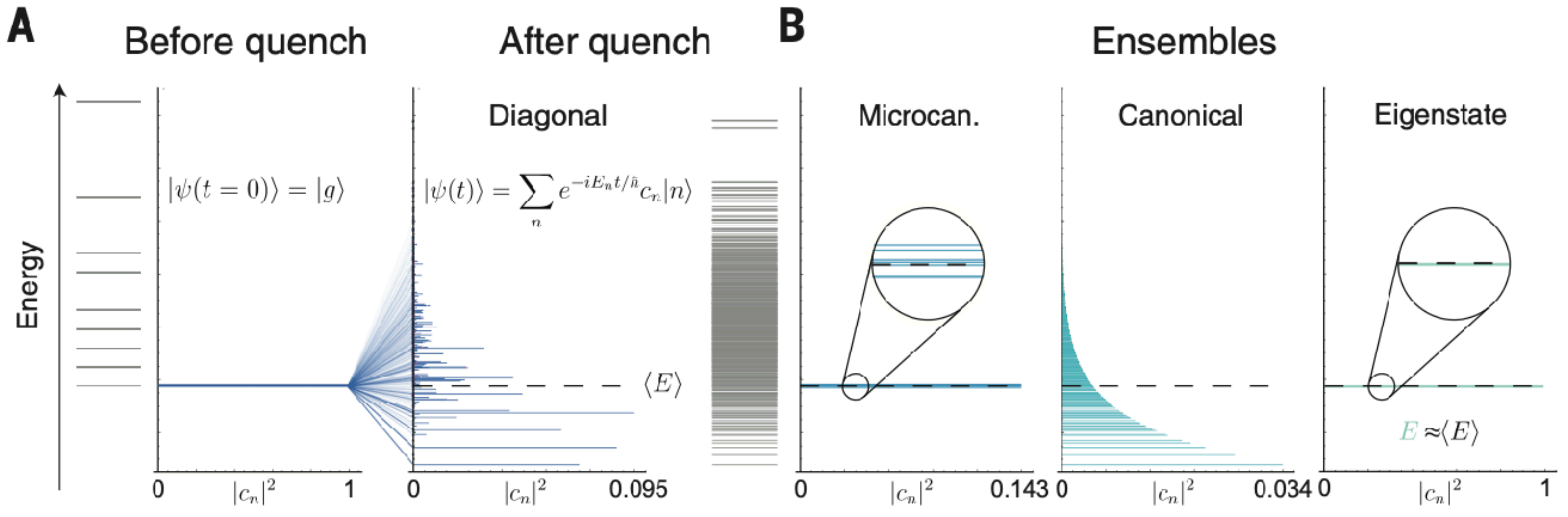
$$F = {\rm Tr}\left[\left(\sqrt{\rho_A^{(T)}}\rho_A\sqrt{\rho_A^{(T)}}\right)^{1/2}\right]$$

When $\rho_A \to \rho_A^{(T)}$, these measures approach the limits $D_{\rm tr} \to 0$ and $F \to 1$.

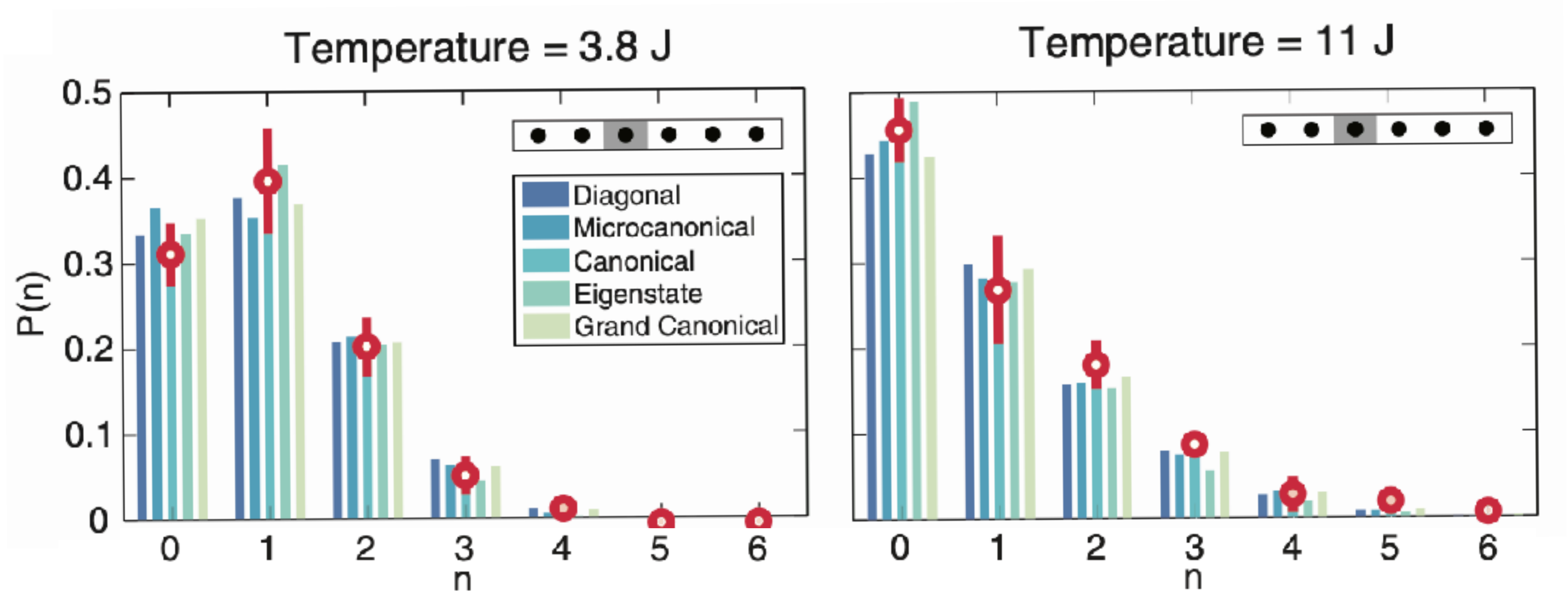
A.M. Kaufman, *et al.*, *Science* 353 (2016) 794



Is the system "thermal"?



After the quench, the initial pure state is projected on many states of the new Hamiltonian, but it remains pure state.



The observed occupation numbers are compared with predictions from several different ensembles. There is remarkable indifference to which ensemble is chosen, even if it is a single energy eigenstate of the new Hamiltonian.

Eigenstate Thermalization Hypothesis (ETH)

Eigenstate Thermalization Hypothesis (ETH) - 1

- Principle: Energy eigenstates of a chaotic quantum system exhibit thermal features.
- Encoded in the matrix elements of an observable *A* in the energy eigenstate basis:

$$A_{\alpha\beta} = \langle E_\alpha | \mathscr{A} | E_\beta \rangle = A(E)\,\delta_{\alpha\beta} + e^{-S(E)/2} f(E,\omega)\, R_{\alpha\beta}$$

where $E = (E_\alpha + E_\beta)/2$ and $\omega = E_\alpha - E_\beta$ and $R_{\alpha\beta}$ is a random matrix.

$S(E)$ is the thermodynamic entropy [M. Srednicki (1999)].

- Off-diagonal matrix elements are statistically suppressed.
- The diagonal matrix elements yield the thermal average of the observable:

$$\langle \mathscr{A} \rangle_T = Z(T)^{-1} \int \frac{dE}{E}\, e^{S(E)-E/T}\, A(E) + O(e^{-S/2})$$

Eigenstate Thermalization Hypothesis (ETH) - 2

Under these assumptions many properties can be proven:

- The quantum system is ergodic when viewed through the observable $\mathscr{A}$:
- The long-time average of $\mathscr{A}$ equals the thermal average $\overline{A} = \langle A \rangle_T$.

  Furthermore:
  - The fluctuations of $\overline{A}$ around $\langle A \rangle_T$ are very small, $O(e^{-S})$.
  - The quantum fluctuations of the system are equal to the thermal fluctuations up to O(1/N^2).
  - The time correlation function of finite-time expectation values $\langle A \rangle_t$ obeys a Kubo relation with the function $f(E,\omega)$ as spectral density.
- In other words, the system, when observed through the observable $\mathscr{A}$ behaves very much like a thermal system.
- Current consensus: ETH holds for systems that exhibit (quantum) chaos.

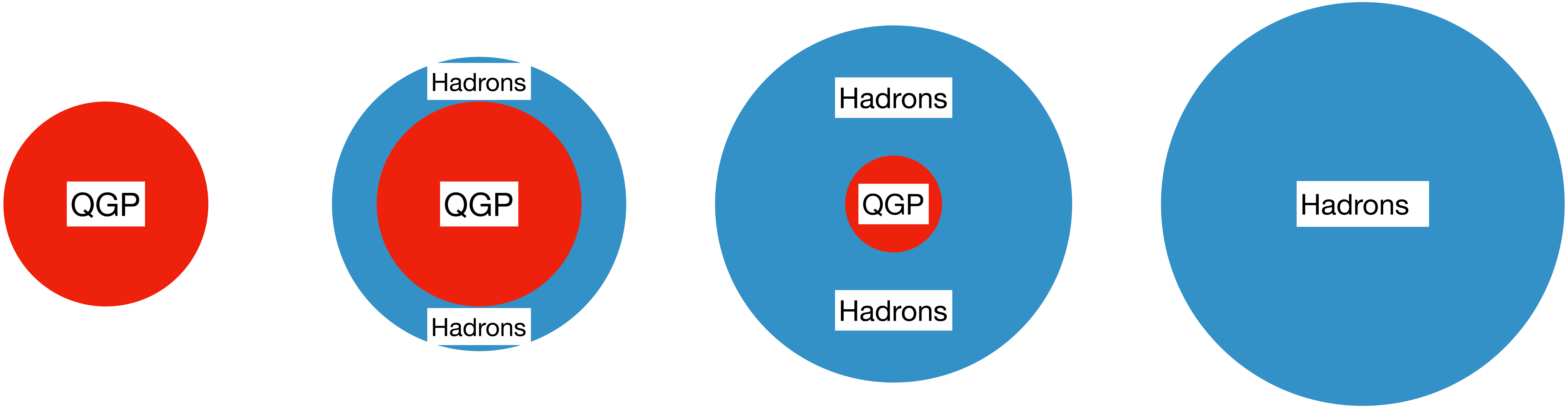
QCD revisited

Several reasons why highly excited systems of quarks and gluons, such as those produced in heavy ion collisions, should appear thermal:

- Yang-Mills SU(N) gauge theories are completely chaotic at the classical level.
- QCD closely resembles gauge theories with a gravity dual: rapid approach to thermal equilibrium can be demonstrated at strong coupling.
- At high energies, the collision of the two nuclei resembles a quench, where empty space between the receding nuclei is subjected to a sudden strong perturbation.
- Experiments observe only fractions of the whole fireball through few-particle observables. The many-body state of emitted hadrons must be highly entangled, but the entanglement is not experimental accessible.
- Different pictures of this thermalization process are complementary.
- They can serve to understand different aspects of it.

QGP Hadronization

QGP Hadronization



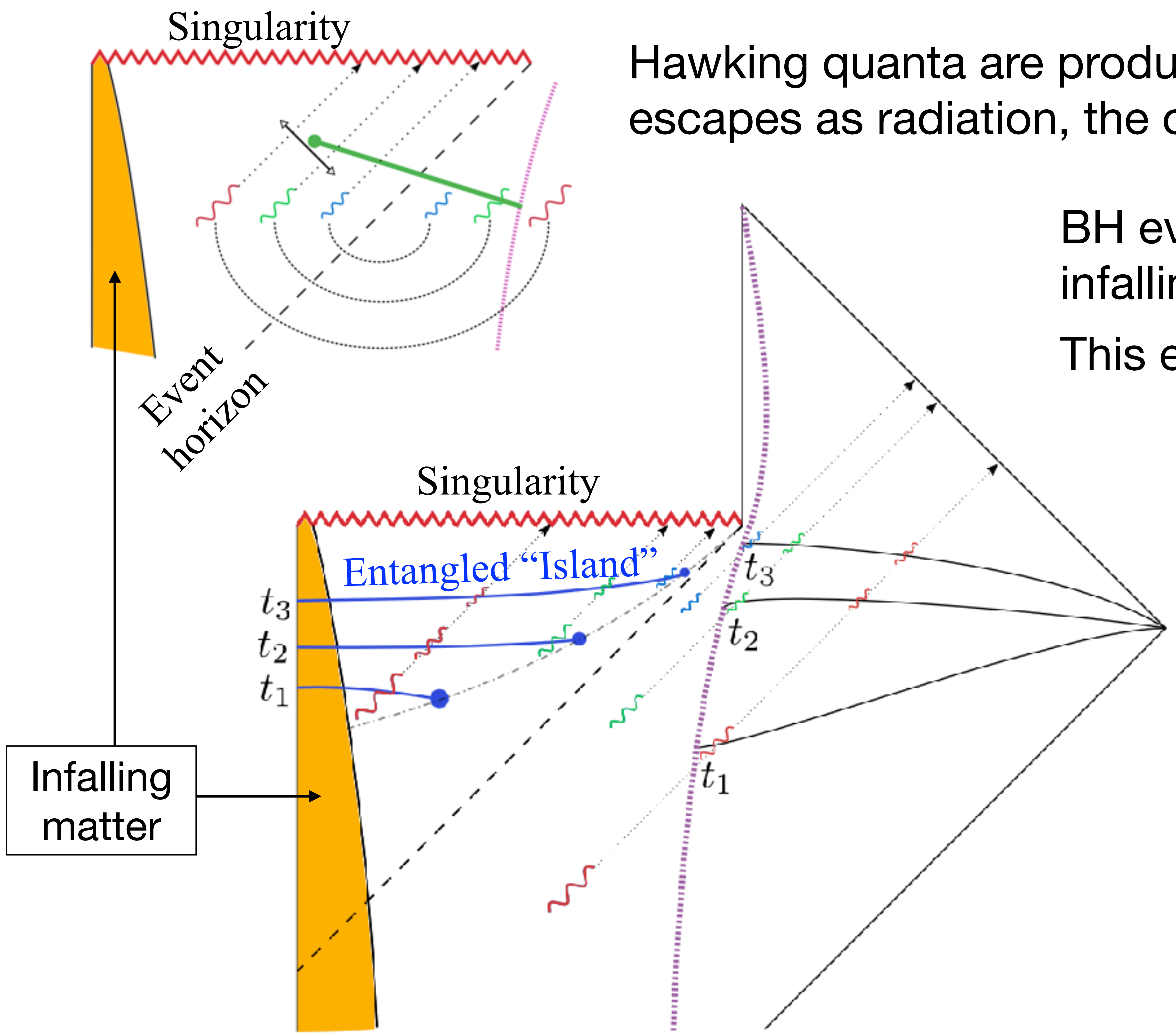
Fully entangled “hot” QGP state

Emitted hadrons mostly entangled with QGP

Emitted hadrons mostly entangled among themselves

When the QGP has evaporated, the hadrons are in a highly entangled pure quantum state that look “thermal” to all practically feasible experiments

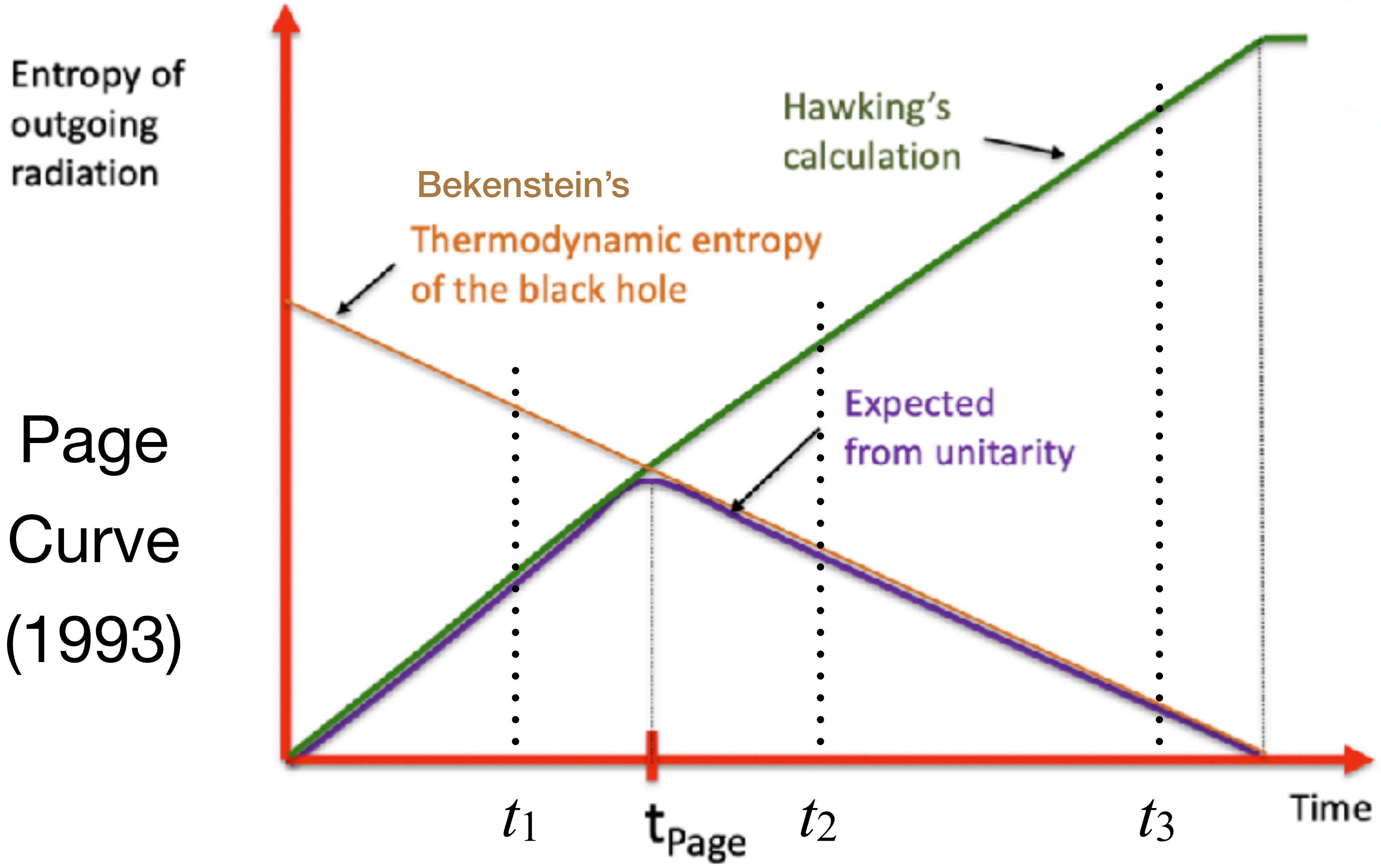
Black hole evaporation



Hawking quanta are produced near the BH event horizon. One quantum escapes as radiation, the other falls into the BH, creating entanglement with BH.

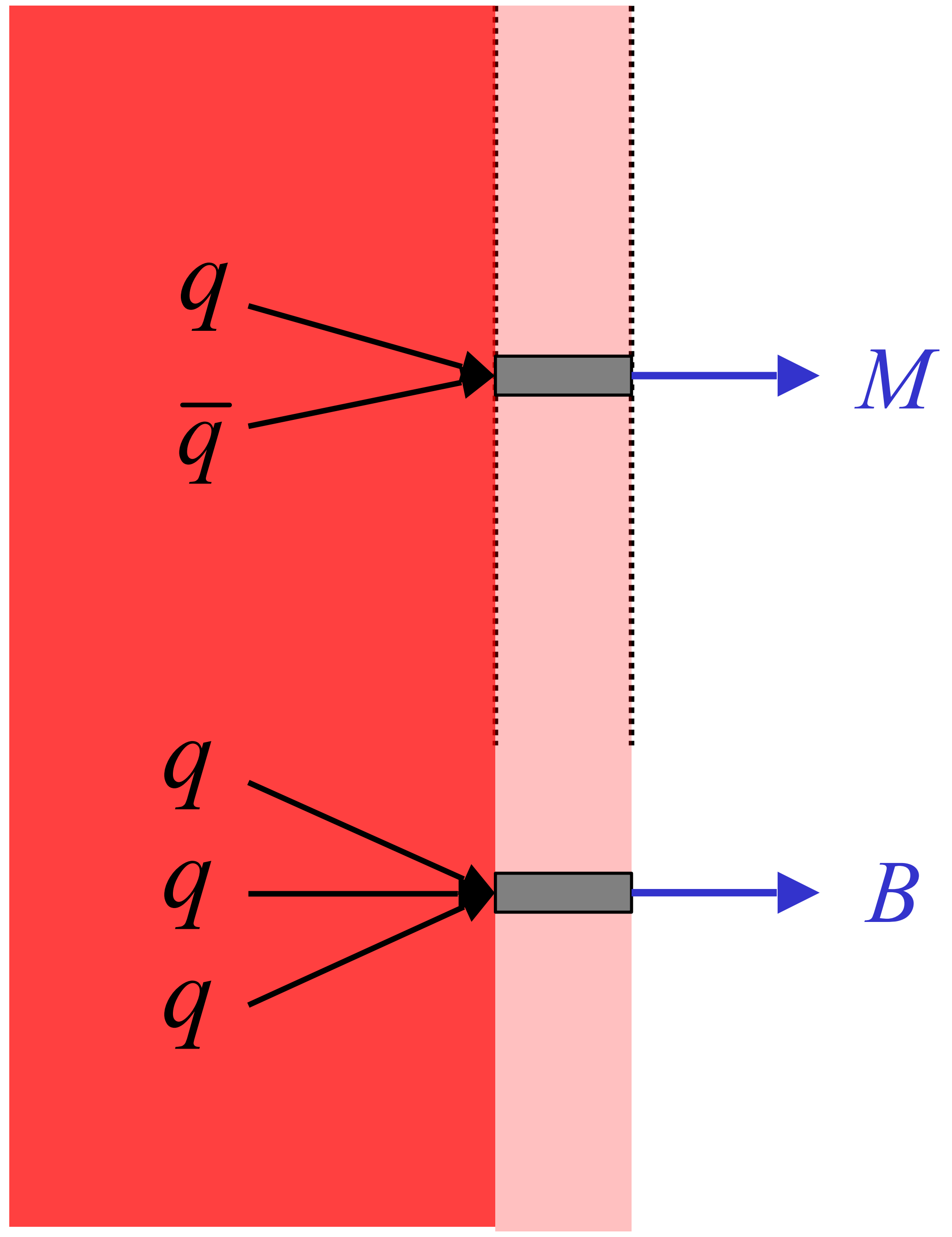
BH evaporation picture benefits from the fact that the infalling quanta propagate ballistically in empty space.

This enables a simple geometric description.



Page Curve (1993)

QGP Evaporation = Hadronization



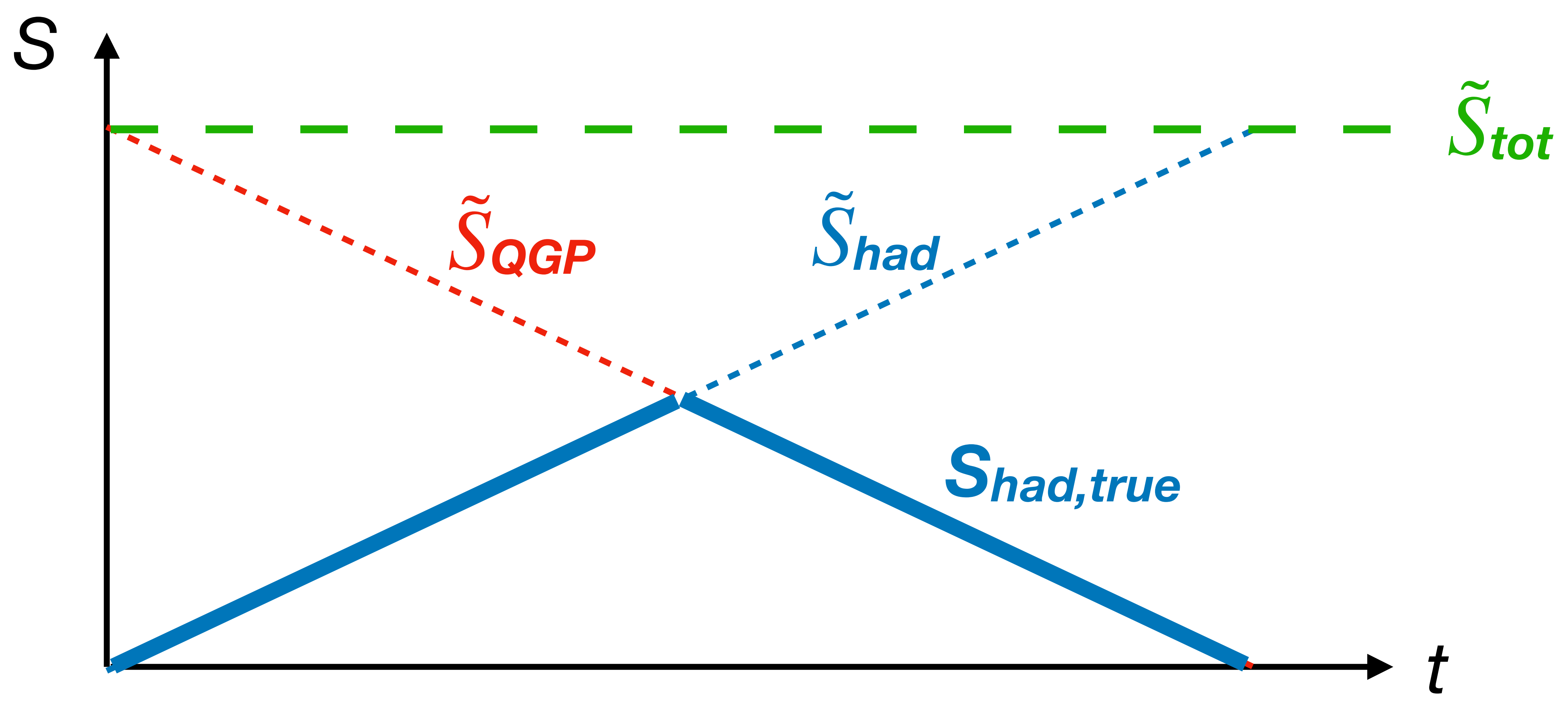
Emission of a meson (baryon) leaves behind a $q\bar{q}$ (qqq) "hole" in the QGP.

There are no such quasi-hole excitations that propagate ballistically and can be traced geometrically.

Requires a direct calculation of the propagation of entanglement entropy.

Alternative: Use holographic dual description, where EE can be calculated geometrically (ongoing research with Joseph Lap and Andreas Schäfer).

QGP "Page Curve"



Summary and Outlook

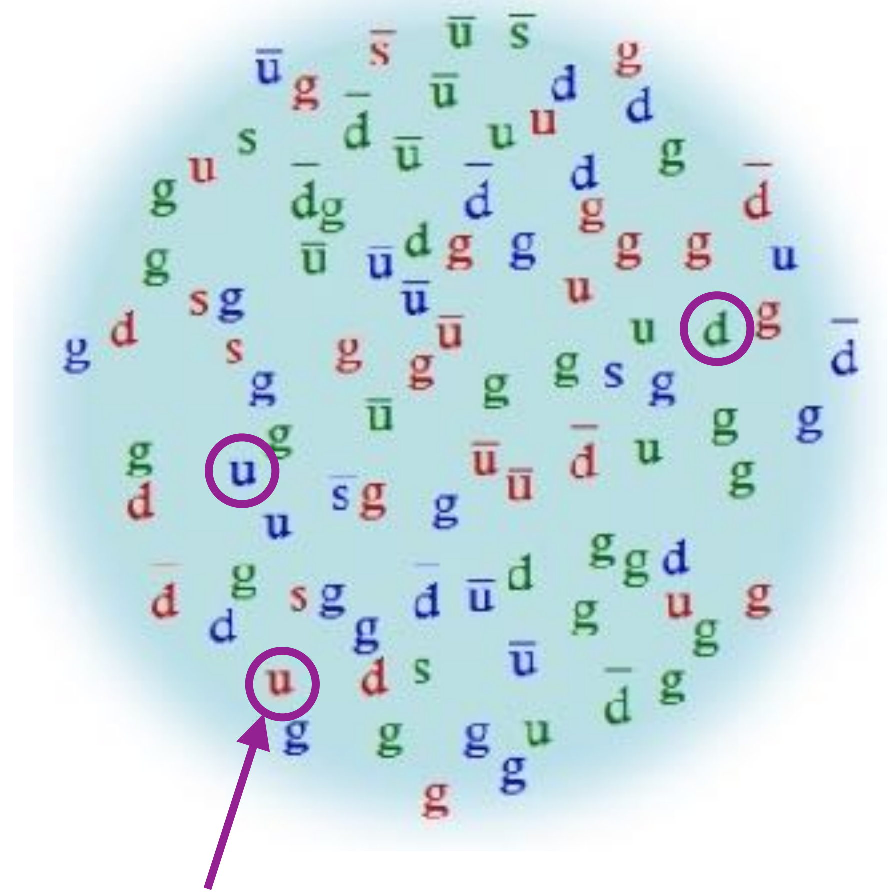
The concepts and insights discussed in this talk have many important applications, e.g.:

- Thermalization and hadronization of the QGP:
  - Simplified by holographic dual picture in QGP-like gauge theories.
- Resolution of the black hole "information paradox”:
  - Where does the information “swallowed” by the BH end up when the BH evaporates?
  - The information is now understood to reside in highly nonlocal entanglement of the emitted “thermal” Hawking radiation with the interior of the BH and ultimately between radiation emitted at early and late times.
- Quantum error correction in quantum information processing:
  - Information lost due to decoherence in the information processing qubits can be retrieved from “surrounding” nodes that store this information via entanglement.
  - Do such error correction schemes scale in practically viable ways as quantum computers grow to contain an ever increasing number of qubits?

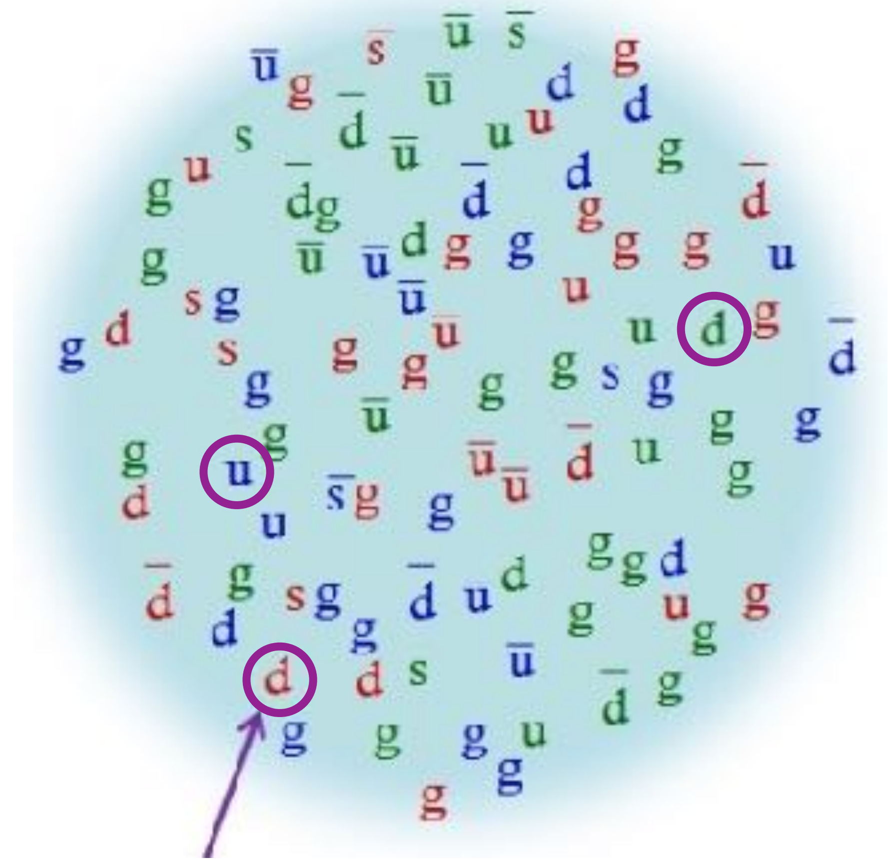
The next frontier

- Quantum structure of the nucleon:
  - Nucleons are highly entangled quantum states of many QCD quanta.
  - Up to now, we have mostly measured single-carton distributions.
  - These measurements entail significant entanglement entropy.
  - Nucleon structure exhibits RG evolution in two variables:
    - Bjorken-x (BFKL)
    - Virtuality Q^2 (DGLAP)
  - How does this relate to entanglement renormalization?
  - How do parton collude to form a color-singlet state with the quantum numbers of the nucleon (spin, isospin, color)?
  - Already being probed in γ+Au/γ+Pb (UPC) at RHIC/LHC.
  - The EIC will allow us to explore such questions in exquisite detail.

proton



neutron



○ Valence quarks - the rest is entangled QCD vacuum