QUALIFYING EXAMINATION, Part 1

Solutions

Problem 1: Mathematical Methods

1.

(a) Upon using $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ...$ valid for $|x| < 1$ we find on setting $n = -1$, $x = a\cos\theta$ (geometric series)

$$I = \int_0^{2\pi} (1 - a\cos\theta + a^2\cos^2\theta + ..)d\theta = 2\pi\left(1 + \frac{a^2}{2}...\right) .$$

(b) Let

$$z = e^{i\theta} \quad \text{so that} \quad d\theta = \frac{dz}{iz} .$$

Then the integral becomes one around the unit circle in the complex plane. Expressing $\cos\theta$ in terms of z and $1/z$, we end up with the contour integral

$$I = \frac{2}{ia}\int_{|z|=1} \frac{dz}{z^2 + 2z/a + 1} .$$

The integrand has two poles at

$$z = -\frac{1}{a} \pm \sqrt{\frac{1}{a^2} - 1} .$$

Clearly one pole is inside the unit circle and one is outside. Using the residue theorem we find

$$I = \frac{2\pi}{\sqrt{1-a^2}}.$$

(c) Expanding this to order a^2

$$I = 2\pi\left(1 + \frac{a^2}{2}...\right) ,$$

we find agreement with the perturbative result in (a).

2. The characteristic equation is

$$(2-\lambda)[(3-\lambda)^2 - 1] = 0 ,$$

with roots

$$\lambda = 4, 2, 2 .$$

For the non-degenerate root we find the (normalized) real eigenvector

$$|4\rangle = \pm\frac{1}{\sqrt{2}}(0,1,-1)^T \ .$$

For the degenerate root if we try

$$|2\rangle = (a,b,c)^T$$

we end up with the condition $b = c$, a arbitrary. Here are two among many possible orthonormal solutions:

$$\pm(1,0,0)^T\,; \qquad \pm\frac{1}{\sqrt{2}}(0,1,1)^T \ .$$

2. Due to the $\frac{1}{x^2}$ in the first bracket, the second has to be expanded to order x^2. Due to the $\frac{1}{x}$ in front of the ln, it has to be expanded to order x^3. Using

$$\frac{1}{1+z} = 1 - z + z^2 + ... \quad \text{and} \quad \ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} + \ldots$$

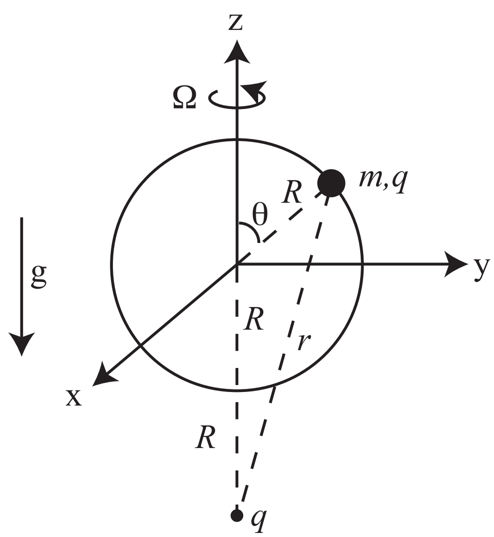
we obtain

$$F(x) = \frac{1+x}{x^2}\left[2(1+x)(1-2x+4x^2+..) - \frac{1}{x}(2x-2x^2+\frac{8x^3}{3}+...)\right] \ ,$$

so that

$$F(0) = \frac{4}{3} \ .$$

Problem 2: Classical Mechanics



(a) The transformation equations are

$$\begin{aligned} x &= R\sin\theta\cos(\Omega t) \\ y &= R\sin\theta\sin(\Omega t) \\ z &= R\cos\theta\ . \end{aligned}$$

(b) The cartesian velocities are

$$\begin{aligned} \dot{x} &= R\dot{\theta}\cos\theta\cos(\Omega t) - R\Omega\sin\theta\sin(\Omega t) \\ \dot{y} &= R\dot{\theta}\cos\theta\sin(\Omega t) + R\Omega\sin\theta\cos(\Omega t) \\ \dot{z} &= -R\dot{\theta}\sin\theta\ . \end{aligned}$$

The kinetic energy is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}mR^2(\dot{\theta}^2 + \Omega^2\sin^2\theta)\ .$$

The potential (gravitational plus electrostatic) is

$$V = mgz + k\frac{q^2}{r} = mgR\cos\theta + k\frac{q^2}{R(5+4\cos\theta)^{1/2}}\ .$$

(c) The Lagrangian is given by

$$L = \frac{1}{2}mR^2(\dot{\theta}^2 + \Omega^2\sin^2\theta) - mgR\cos\theta - k\frac{q^2}{R(5+4\cos\theta)^{1/2}}\ .$$

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}} &= mR^2\dot{\theta} \\
\frac{\partial L}{\partial \theta} &= mR^2 \sin\theta \left(\Omega^2 \cos\theta + \frac{g}{R}\right) - \frac{2kq^2}{R} \frac{\sin\theta}{(5+4\cos\theta)^{3/2}} \ .
\end{aligned}
$$

The Euler-Lagrange equation of motion is

$$
\ddot{\theta} = \sin\theta \left(\Omega^2 \cos\theta + \frac{g}{R}\right) - 2\lambda \frac{g}{R} \frac{\sin\theta}{(5+4\cos\theta)^{3/2}} \ .
$$

(d) At equilibrium $\ddot{\theta} = \dot{\theta} = 0$ and the r.h.s. of the above equation vanishes. Thus either $\sin\theta_0 = 0$ (i.e., $\theta = 0, \pi$) or

$$
\Omega^2 \cos\theta_0 + \frac{g}{R} - 2\lambda \frac{g}{R} \frac{1}{(5+4\cos\theta_0)^{3/2}} = 0 \ .
$$

Using the special value $\Omega^2 = 4g/5R$, we find

$$
\cos\theta_0 = \frac{1}{4}[(10\lambda)^{2/5} - 5] \ .
$$

This solution exists when $|\cos\theta_0| \leq 1$, i.e.,

$$
1 \leq (10\lambda)^{2/5} \leq 9 \ .
$$

(e) The conjugate momentum is

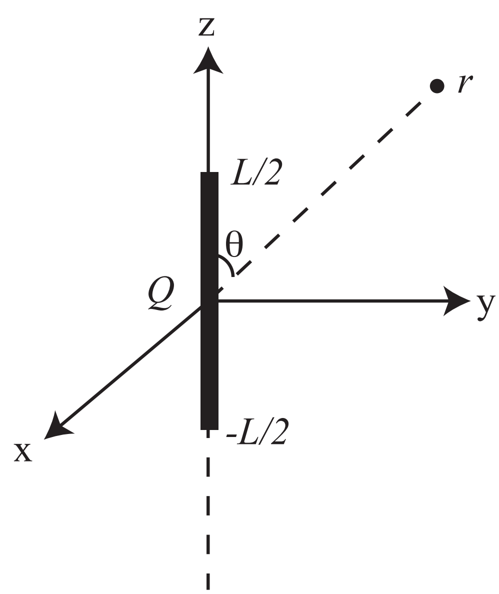
$$
p_\theta = \frac{\partial L}{\partial \theta} = mR^2\dot{\theta} \ .
$$

Thus the Hamiltonian is

$$
\begin{aligned}
H &= p_\theta \dot{\theta} - L = \frac{1}{2} mR^2 (\dot{\theta}^2 - \Omega^2 \sin^2\theta) + mgR\cos\theta + k\frac{q^2}{R(5+4\cos\theta)^{1/2}} \\
&= \frac{p_\theta^2}{2mR^2} - \frac{1}{2} mR^2 \Omega^2 \sin^2\theta + mgR\cos\theta + k\frac{q^2}{R(5+4\cos\theta)^{1/2}} \ .
\end{aligned}
$$

The Hamiltonian is conserved since it does not depend explicitly on time and $dH/dt = \partial H/\partial t = 0$. The Hamiltonian is not equal to the energy since the transformation equations depend explicitly on time (see part (a)).

Problem 3: Electromagnetism I



(a) The potential along the z-axis may be computed by integrating the contributions of point charges along the rod:

$$\Phi(z) = \int_{-L/2}^{L/2} \frac{Q}{L}\frac{1}{z-\xi}\mathrm{d}\xi = -\frac{Q}{L}\ln(z-\xi)\Big|_{-L/2}^{L/2}$$

$$= -\frac{Q}{L}\ln\left(\frac{z-L/2}{z+L/2}\right) = \frac{Q}{L}\ln\left(\frac{z+L/2}{z-L/2}\right) = \frac{Q}{L}\ln\left(\frac{1+u}{1-u}\right) ,$$

where $u \equiv L/(2z)$.

(b) Since we consider $r > L/2$, all the A_l are zero, and

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) .$$

We wish to equate this with the result found in (a) along the z-axis where $\theta = 0$ and $P_l(1) = 1$ for all l. We must rewrite $\ln[(1+u)/(1-u)]$ as a sum. Now

$$\ln(1+u) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}u^n}{n}.$$

So

$$\ln\left(\frac{1+u}{1-u}\right) = \sum_{n=1}^{\infty}\left[(-1)^{n+1} - (-1)^{n+1}(-1)^n\right]\frac{u^n}{n} = 2\sum_{l=0}^{\infty}\frac{u^{2l+1}}{2l+1} .$$

Along the z axis, $u = L/(2z) = L/(2r)$ and the potential is

$$\Phi(r,0) = \frac{2Q}{L}\sum_{l=0}^{\infty}\frac{1}{2l+1}\left(\frac{L}{2}\right)^{2l+1}\frac{1}{r^{2l+1}} = Q\sum_{l=0}^{\infty}\frac{1}{2l+1}\left(\frac{L}{2}\right)^{2l}\frac{1}{r^{2l+1}} \ .$$

Comparing with the expansion in Legendre polynomials for $\theta = 0$, we find

$$B_{2l} = Q\frac{1}{2l+1}\left(\frac{L}{2}\right)^{2l} \quad ; \quad B_{2l+1} = 0 \ .$$

Thus the potential is given by

$$\Phi(r,\theta) = Q\sum_{l=0}^{\infty}\frac{1}{2l+1}\left(\frac{L}{2}\right)^{2l}\frac{1}{r^{2l+1}}P_{2l}(\cos\theta) \ .$$

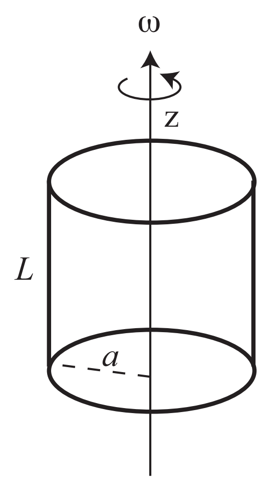
(c) For $r \gg L$, the dominating term is $l = 0$ (all other terms are suppressed by powers of $L/r \ll 1$). This gives

$$\Phi(r,\theta) = \frac{Q}{r} \ .$$

which is appropriate for charged rods, "which from far off look like flies" [Foucault], i.e., from far away the charged rod looks like a point charge.

Problem 4: Electromagnetism II

1.



(a) The field can be calculated by dividing the surface into infinitesimal current loops carrying currents $dI = \omega a \sigma dz'$, and adding their fields which is a function of their position relative to the observation point.

The Biot-Savart law can be used to determine the field dB_z of a single current loop, which along the symmetry axis of the loop is directed along z (in using the Biot-Savart law the contribution from each segment of the loop has to be projected on the z axis). At a distance z'' from the center of the loop, we find

$$dB_z = \frac{\mu_0}{2} \frac{a^2 dI}{(a^2 + z''^2)^{3/2}} \ .$$

To find the total magnetic field along the cylinder axis at a distance z from the center, we integrate the contributions of the rings at distance z' from the center. Using $z'' = z - z'$, we have

$$B_z = \frac{\mu_0}{2} \int_{-L/2}^{L/2} \omega a \sigma \frac{a^2 dz'}{[(z - z')^2 + R^2]^{3/2}} \ .$$

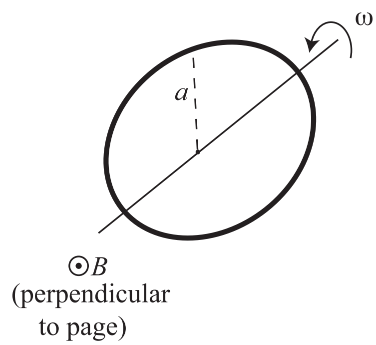
Using the integral given in the hint, we find:

$$B_z = \frac{\mu_0 \omega \sigma a}{2} \left[\frac{z + L/2}{\sqrt{a^2 + (z + L/2)^2}} - \frac{z - L/2}{\sqrt{a^2 + (z - L/2)^2}} \right] \ .$$

(b) For large z, the magnetic fields of the cylinder and of the ring are determined by their magnetic dipole moment alone. The latter is proportional to IA (the product of the area of and the total current). For the cylinder $I = \int dI = \omega a \sigma L$, and $A = \pi a^2$ so any loop for which $IA = (\pi a^2)(\omega a \sigma L)$ will produce the same (long-distance) dipole field.

(c) The current and charge density are stationary so there is no radiation.

2.



(a) Initially $E = \mathcal{I}\omega_0^2/2$.

(b) The EMF around the loop is given by $V = d\Phi_B/dt = \omega\pi a^2 B\cos(\omega t + \phi)$ where ϕ is the initial phase. The current is equal to $I = V/R$, and the power dissipated is $IV = \frac{V^2}{R}$, so

$$\frac{dE}{dt} = -\frac{V^2}{R} = -\frac{[\omega\pi a^2 B\cos(\omega t + \phi)]^2}{R} \ .$$

(c) In this limit, we can replace the $\cos^2$ factor by its average over a cycle, which is 1/2. Also, using $E = \mathcal{I}\omega^2/2$, we have $\frac{dE}{dt} = \mathcal{I}\omega\frac{d\omega}{dt}$. The differential equation is then

$$\mathcal{I}\omega\frac{d\omega}{dt} = -\frac{(\omega\pi a^2 B)^2}{2R} \ ,$$

which leads to

$$\frac{d\omega}{dt} = -\frac{\pi^2 a^4 B^2}{2R\mathcal{I}}\omega \equiv -\gamma\omega \ .$$

The solution is $\omega = \omega_0 e^{-\gamma t}$. The exponential damping time is $\tau = 1/\gamma = \frac{2R\mathcal{I}}{\pi^2 a^4 B^2}$, and in one time constant τ, the angular velocity falls to ω_0/e.