Solutions

Problem 1: Mathematical Methods

1. (a) Upon using \((1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots\) valid for \(|x| < 1\) we find on setting \(n = -1, x = a \cos \theta\) (geometric series)

\[
I = \int_0^{2\pi} (1 - a \cos \theta + a^2 \cos^2 \theta + ..)d\theta = 2\pi \left( 1 + \frac{a^2}{2} \ldots \right).
\]

(b) Let

\[z = e^{i\theta}\] so that \(d\theta = \frac{dz}{iz} \).

Then the integral becomes one around the unit circle in the complex plane. Expressing \(\cos \theta\) in terms of \(z\) and \(1/z\), we end up with the contour integral

\[
I = \frac{2}{ia} \int_{|z|=1} \frac{dz}{z^2 + 2z/a + 1}.
\]

The integrand has two poles at

\[z = -\frac{1}{a} \pm \sqrt{\frac{1}{a^2} - 1} \,.
\]

Clearly one pole is inside the unit circle and one is outside. Using the residue theorem we find

\[
I = \frac{2\pi}{\sqrt{1 - a^2}}.
\]

(c) Expanding this to order \(a^2\)

\[
I = 2\pi \left( 1 + \frac{a^2}{2} \ldots \right),
\]

we find agreement with the perturbative result in (a).

2. The characteristic equation is

\[
(2 - \lambda)[(3 - \lambda)^2 - 1] = 0,
\]

with roots

\[
\lambda = 4, 2, 2.
\]
For the non-degenerate root we find the (normalized) real eigenvector

\[ |4\rangle = \pm \frac{1}{\sqrt{2}} (0, 1, -1)^T. \]

For the degenerate root if we try

\[ |2\rangle = (a, b, c)^T \]

we end up with the condition \( b = c, \ a \) arbitrary. Here are two among many possible orthonormal solutions:

\[ \pm (1, 0, 0)^T; \pm \frac{1}{\sqrt{2}} (0, 1, 1)^T. \]

2. Due to the \( \frac{1}{x^2} \) in the first bracket, the second has to be expanded to order \( x^2 \). Due to the \( \frac{1}{x} \) in front of the \( \ln \), it has to be expanded to order \( x^3 \). Using

\[
\frac{1}{1 + z} = 1 - z + z^2 + \ldots \quad \text{and} \quad \ln(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} + \ldots
\]

we obtain

\[
F(x) = \frac{1 + x}{x^2} \left[ 2(1 + x)(1 - 2x + 4x^2 + \ldots) - \frac{1}{x}(2x - 2x^2 + \frac{8x^3}{3} + \ldots) \right],
\]

so that

\[
F(0) = \frac{4}{3}.
\]
(a) The transformation equations are

\[ x = R \sin \theta \cos(\Omega t) \]
\[ y = R \sin \theta \sin(\Omega t) \]
\[ z = R \cos \theta . \]

(b) The cartesian velocities are

\[ \dot{x} = R\dot{\theta} \cos \theta \cos(\Omega t) - R\Omega \sin \theta \sin(\Omega t) \]
\[ \dot{y} = R\dot{\theta} \cos \theta \sin(\Omega t) + R\Omega \sin \theta \cos(\Omega t) \]
\[ \dot{z} = -R\dot{\theta} \sin \theta . \]

The kinetic energy is

\[ T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2} m R^2 (\dot{\theta}^2 + \Omega^2 \sin^2 \theta) . \]

The potential (gravitational plus electrostatic) is

\[ V = mgz + k \frac{q^2}{r} = mgR \cos \theta + k \frac{q^2}{R(5 + 4 \cos \theta)^{1/2}} . \]

(c) The Lagrangian is given by

\[ L = \frac{1}{2} m R^2 (\dot{\theta}^2 + \Omega^2 \sin^2 \theta) - mgR \cos \theta - k \frac{q^2}{R(5 + 4 \cos \theta)^{1/2}} . \]
\[
\begin{align*}
\frac{\partial L}{\partial \dot{\theta}} &= mR^2 \dot{\theta} \\
\frac{\partial L}{\partial \theta} &= mR^2 \sin \theta \left( \Omega^2 \cos \theta + \frac{g}{R} \right) - \frac{2kq^2}{R} \frac{\sin \theta}{(5 + 4 \cos \theta)^{3/2}}.
\end{align*}
\]

The Euler-Lagrange equation of motion is
\[
\ddot{\theta} = \sin \theta \left( \Omega^2 \cos \theta + \frac{g}{R} \right) - 2\lambda \frac{g}{R} \frac{\sin \theta}{(5 + 4 \cos \theta)^{3/2}}.
\]

(d) At equilibrium \( \ddot{\theta} = \dot{\theta} = 0 \) and the r.h.s. of the above equation vanishes. Thus either \( \sin \theta_0 = 0 \) (i.e., \( \theta = 0, \pi \)) or
\[
\Omega^2 \cos \theta_0 + \frac{g}{R} - 2\lambda \frac{g}{R} \frac{1}{(5 + 4 \cos \theta_0)^{3/2}} = 0.
\]

Using the special value \( \Omega^2 = 4g/5R \), we find
\[
\cos \theta_0 = \frac{1}{4}[(10\lambda)^{2/5} - 5].
\]

This solution exists when \( |\cos \theta_0| \leq 1 \), i.e.,
\[
1 \leq (10\lambda)^{2/5} \leq 9.
\]

(e) The conjugate momentum is
\[
p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mR^2 \dot{\theta}.
\]

Thus the Hamiltonian is
\[
H = p_\theta \dot{\theta} - L = \frac{1}{2} mR^2 (\dot{\theta}^2 - \Omega^2 \sin^2 \theta) + mgR \cos \theta + k \frac{q^2}{R(5 + 4 \cos \theta)^{1/2}}
\]
\[
= \frac{p_\theta^2}{2mR^2} - \frac{1}{2} mR^2 \Omega^2 \sin^2 \theta + mgR \cos \theta + k \frac{q^2}{R(5 + 4 \cos \theta)^{1/2}}.
\]

The Hamiltonian is conserved since it does not depend explicitly on time and \( dH/dt = \partial H/\partial t = 0 \). The Hamiltonian is not equal to the energy since the transformation equations depend explicitly on time (see part (a)).
Problem 3: Electromagnetism I

(a) The potential along the $z$-axis may be computed by integrating the contributions of point charges along the rod:

$$\Phi(z) = \int_{-L/2}^{L/2} \frac{Q}{L} \frac{1}{z - \xi} \, d\xi = - \frac{Q}{L} \ln |z - \xi| \bigg|_{-L/2}^{L/2}$$

$$= -\frac{Q}{L} \ln \left( \frac{z - L/2}{z + L/2} \right) = \frac{Q}{L} \ln \left( \frac{z + L/2}{z - L/2} \right) = \frac{Q}{L} \ln \left( \frac{1 + u}{1 - u} \right),$$

where $u \equiv L/(2z)$.

(b) Since we consider $r > L/2$, all the $A_l$ are zero, and

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta).$$

We wish to equate this with the result found in (a) along the $z$-axis where $\theta = 0$ and $P_l(1) = 1$ for all $l$. We must rewrite $\ln[(1 + u)/(1 - u)]$ as a sum. Now

$$\ln(1 + u) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} u^n}{n}.$$

So

$$\ln \left( \frac{1 + u}{1 - u} \right) = \sum_{n=1}^{\infty} \left[ (-1)^{n+1} - (-1)^{n+1}(-1)^n \right] \frac{u^n}{n} = 2 \sum_{l=0}^{\infty} \frac{u^{2l+1}}{2l + 1}.$$
Along the \( z \) axis, \( u = L/(2z) = L/(2r) \) and the potential is

\[
\Phi(r, 0) = \frac{2Q}{L} \sum_{l=0}^{\infty} \frac{1}{2l + 1} \left( \frac{L}{2} \right)^{2l+1} \frac{1}{r^{2l+1}} = Q \sum_{l=0}^{\infty} \frac{1}{2l + 1} \left( \frac{L}{2} \right)^{2l} \frac{1}{r^{2l+1}}.
\]

Comparing with the expansion in Legendre polynomials for \( \theta = 0 \), we find

\[
B_{2l} = Q \frac{1}{2l + 1} \left( \frac{L}{2} \right)^{2l} ; \quad B_{2l+1} = 0.
\]

Thus the potential is given by

\[
\Phi(r, \theta) = Q \sum_{l=0}^{\infty} \frac{1}{2l + 1} \left( \frac{L}{2} \right)^{2l} \frac{1}{r^{2l+1}} P_{2l}(\cos \theta).
\]

(c) For \( r \gg L \), the dominating term is \( l = 0 \) (all other terms are suppressed by powers of \( L/r \ll 1 \)). This gives

\[
\Phi(r, \theta) = \frac{Q}{r}.
\]

which is appropriate for charged rods, “which from far off look like flies” [Foucault], i.e., from far away the charged rod looks like a point charge.
(a) The field can be calculated by dividing the surface into infinitesimal current loops carrying currents $dI = \omega a \sigma dz'$, and adding their fields which is a function of their position relative to the observation point.

The Biot-Savart law can be used to determine the field $dB_z$ of a single current loop, which along the symmetry axis of the loop is directed along $z$ (in using the Biot-Savart law the contribution from each segment of the loop has to be projected on the $z$ axis). At a distance $z''$ from the center of the loop, we find

$$dB_z = \frac{\mu_0}{2} \frac{a^2 dI}{(a^2 + z''^2)^{3/2}}.$$

To find the total magnetic field along the cylinder axis at a distance $z$ from the center, we integrate the contributions of the rings at distance $z'$ from the center. Using $z'' = z - z'$, we have

$$B_z = \frac{\mu_0}{2} \int_{-L/2}^{L/2} \omega a \sigma \frac{a^2 dz'}{[(z - z')^2 + R^2]^{3/2}}.$$

Using the integral given in the hint, we find:

$$B_z = \frac{\mu_0 \omega \sigma a}{2} \left[ \frac{z + L/2}{\sqrt{a^2 + (z + L/2)^2}} - \frac{z - L/2}{\sqrt{a^2 + (z - L/2)^2}} \right].$$
(b) For large $z$, the magnetic fields of the cylinder and of the ring are determined by their magnetic dipole moment alone. The latter is proportional to $IA$ (the product of the area of and the total current). For the cylinder $I = \int dI = \omega a \sigma L$, and $A = \pi a^2$ so any loop for which $IA = (\pi a^2)(\omega a \sigma L)$ will produce the same (long-distance) dipole field.

(c) The current and charge density are stationary so there is no radiation.

2.

\[ \omega \]

\[ B \]

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(a) Initially $E = \mathcal{I} \omega_0^2/2$.

(b) The EMF around the loop is given by $V = d\Phi_B/dt = \omega \pi a^2 B \cos(\omega t + \phi)$ where $\phi$ is the initial phase. The current is equal to $I = V/R$, and the power dissipated is $IV = \frac{V^2}{R}$, so

\[ \frac{dE}{dt} = -\frac{V^2}{R} = -\frac{[\omega \pi a^2 B \cos(\omega t + \phi)]^2}{R}. \]

(c) In this limit, we can replace the $\cos^2$ factor by its average over a cycle, which is $1/2$. Also, using $E = \mathcal{I} \omega^2/2$, we have $\frac{dE}{dt} = \mathcal{I} \omega \frac{d\omega}{dt}$. The differential equation is then

\[ \mathcal{I} \omega \frac{d\omega}{dt} = -\frac{(\omega \pi a^2 B)^2}{2R}, \]

which leads to

\[ \frac{d\omega}{dt} = -\frac{\pi^2 a^4 B^2}{2 R \mathcal{I}} \omega \equiv -\gamma \omega. \]

The solution is $\omega = \omega_0 e^{-\gamma t}$. The exponential damping time is $\tau = 1/\gamma = \frac{2 R \mathcal{I}}{\pi^2 a^4 B^2}$, and in one time constant $\tau$, the angular velocity falls to $\omega_0/e$. 

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