QUALIFYING EXAMINATION, Part 2

Solutions

Problem 1: Quantum Mechanics I

1. (a) Using the representation of $(\hat{S}_x, \hat{S}_y, \hat{S}_z)$ in terms of Pauli matrices, we have

$$S_{z'} = \frac{\hbar}{2} \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}.$$

It is simple to check that the 2 eigenvalues are $\pm\hbar/2$. Now, we put

$$|z';+\rangle = \alpha|+\rangle + \beta|-\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

The equation $S_{z'}|z';+\rangle = +\frac{\hbar}{2}|z';+\rangle$ gives:

$$\alpha = \cos\theta\alpha + \sin\theta e^{-i\phi}\beta$$
$$\beta = \sin\theta e^{i\phi}\alpha - \cos\theta\beta.$$

We redefine $\alpha = e^{-i\phi/2}\tilde{\alpha}$ and $\beta = e^{i\phi/2}\tilde{\beta}$ such that:

$$\tilde{\alpha} = \cos\theta\tilde{\alpha} + \sin\theta\tilde{\beta}$$
$$\tilde{\beta} = \sin\theta\tilde{\alpha} - \cos\theta\tilde{\beta}.$$

Therefore,

$$\tilde{\alpha} = \cos(\theta/2)e^{i\lambda}$$
$$\tilde{\beta} = \sin(\theta/2)e^{i\lambda},$$

where λ is an overall (arbitrary) phase. For simplicity here we set $\lambda = 0$. Then:

$$|z';+\rangle = \cos(\theta/2)e^{-i\frac{\phi}{2}}|+\rangle + \sin(\theta/2)e^{i\frac{\phi}{2}}|-\rangle.$$

(For $\hat{z}' = \hat{x}$, we check that $|x;+\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ and the eigenstate is defined modulo an overall phase.)

Another solution (Y.A.): we can find $|z';+\rangle$ by rotating $|+\rangle$ to the $\hat{z}'$ direction. The Euler angles of the corresponding rotation $\hat{R}$ are $(\alpha,\beta,\gamma)=(\phi,\theta,0)$ so that $\hat{R}=e^{-i\alpha\hat{S}_z}e^{-i\beta\hat{S}_y}e^{-i\gamma\hat{S}_z}=e^{-i\phi\hat{S}_z}e^{-i\theta\hat{S}_y}$. Then

$$
\begin{aligned}
|z';+\rangle = e^{-i\phi\hat{S}_z}e^{-i\theta\hat{S}_y}|+\rangle &= e^{-i\phi\hat{S}_z}[\cos(\theta/2)|+\rangle+\sin(\theta/2)|-\rangle] \\
&= e^{-i\frac{\phi}{2}}\cos(\theta/2)|+\rangle + e^{i\frac{\phi}{2}}\sin(\theta/2)|-\rangle
\end{aligned}
$$

(b)

$$|\langle+|z';+\rangle|^2=\cos^2(\theta/2) \text{ and } |\langle-|z';+\rangle|^2=\sin^2(\theta/2).$$

2. (a) The Hamiltonian reads:

$$H=\tfrac{1}{2m}\left[\left(p_x+\tfrac{eB}{c}y\right)^2+p_y^2+p_z^2\right].$$

Formally, an obervable $\hat{Q}$ that does not depend on time explicitly is a constant of motion if $\hat{Q}$ commutes with the Hamiltonian. We check that

$$[\hat{p}_x,H]=0 \text{ and } [\hat{p}_z,H]=0.$$

(b)

$$
\begin{aligned}
\tfrac{1}{2m}\left[(p_x+\tfrac{eB}{c}y)^2-\hbar^2\tfrac{d^2}{dy^2}+p_z^2\right]\phi(y) &= E\phi(y) \\
-\tfrac{\hbar^2}{2m}\tfrac{d^2\phi}{dy^2}+\tfrac{m}{2}\left(\tfrac{eB}{mc}\right)^2(y+\tfrac{cp_x}{eB})^2\phi(y) &= \left(E-\tfrac{p_z^2}{2m}\right)\phi(y).
\end{aligned}
$$

We set

$$\omega=\tfrac{|e|B}{mc} \qquad y'=y+\tfrac{cp_x}{eB} \qquad E'=E-p_z^2/2m.$$

The equation becomes:

$$-\tfrac{\hbar^2}{2m}\tfrac{d^2\phi}{dy'^2}+\tfrac{m}{2}\omega^2y'^2\phi=E'\phi.$$

This is the eigen-equation for a 1D harmonic oscillator. The energy eigenvalues are therefore:

$$E_n=\tfrac{p_z^2}{2m}+(n+\tfrac{1}{2})\hbar\omega.$$

Classical interpretation: The motion parallel to the magnetic field is not coupled to the transverse motion, and is unaffected by the field. The motion in the plane perpendicular to the field is in a circular orbit with angular frequency $\omega=|e|B/(mc)$ (the cyclotron frequency), and these periodic orbits correspond to discrete energy levels whose separation is $\hbar\omega$.

Problem 2: Quantum Mechanics II

1. (a) Introducing creation and annihilation operators as in the hint, we have

$$H_0/\hbar\omega = \left(a^\dagger a + \frac{1}{2}\right) ,$$

with eigenstates and eigenenergies

$$\begin{aligned} |n\rangle &= \frac{1}{\sqrt{n!}}(a^\dagger)^n |0\rangle \\ E_n &= \left(n + \frac{1}{2}\right)\hbar\omega. \end{aligned}$$

The perturbation V can be written as

$$V = -qb\left(\frac{\hbar}{2m\omega}\right)^{1/2}\left(a^\dagger + a\right).$$

In first order perturbation theory

$$\langle n | V | n\rangle = 0$$

In second order perturbation theory

$$\Delta E_n = \sum_{n'} \frac{\langle n | V | n'\rangle \langle n' | V | n\rangle}{E_n - E_{n'}} = \sum_{n'} \frac{|\langle n | V | n'\rangle|^2}{E_n - E_{n'}}.$$

But

$$\begin{aligned} a^\dagger |n\rangle &= \sqrt{n+1}\,|n+1\rangle \\ a\,|n\rangle &= \sqrt{n}\,|n-1\rangle . \end{aligned}$$

Thus $n' = n \pm 1$, and

$$\Delta E_n = q^2 b^2 \frac{\hbar}{m\omega}\frac{1}{2}[n - (n+1)]\frac{1}{\hbar\omega} = -\frac{q^2 b^2}{2m\omega^2} .$$

(b) The Hamiltonian problem with

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 x^2 - qbx$$

can be exactly solved by rewriting H as

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\left(x - \frac{qb}{m\omega^2}\right)^2 - \frac{q^2b^2}{2m\omega^2}$$

with eigenenergies

$$\left(n+\frac{1}{2}\right)\hbar\omega - \frac{q^2b^2}{2m\omega^2}.$$

Thus the exact result and 2nd order perturbation theory coincide.

2. (a) The trial wave function is spherically symmetric (it depends only on r) and therefore has good angular momentum $l=0$.

(b) The energy functional is

$$E(b) = \frac{\langle\Phi(b)\,|H|\,\Phi(b)\rangle}{\langle\Phi(b)\mid\Phi(b)\rangle}\ .$$

We use the Hamitlonian

$$H = \frac{\hat{p}^2}{2m} - \frac{e^2}{r}$$

with $\hat{p}^2 = \hat{p}_r^2 + \vec{l}^2/r^2$. Since Φ has good angular momentum $l=0$, we find

$$\begin{aligned}
\langle\Phi\mid\Phi\rangle &= \int_0^\infty r^2 dr\,|u_0(r)|^2 \\
\langle\Phi\,|H|\,\Phi\rangle &= \int_0^\infty r^2 dr u_0(r)\left[-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} - \frac{e^2}{r}\right]u_0(r).
\end{aligned}$$

Using $u_0(r) = e^{-br}$

$$\begin{aligned}
\langle\Phi\mid\Phi\rangle &= \int_0^\infty r^2 e^{-2br} dr \\
\langle\Phi\,|H|\,\Phi\rangle &= \int_0^\infty r^2 e^{-br}\left[-\frac{\hbar^2}{2m}\left(-\frac{2b}{r}e^{-br} + b^2 e^{-br}\right) - \frac{e^2}{r}e^{-br}\right]dr.
\end{aligned}$$

Using the integral in the hint we obtain

$$E(b) = \frac{\langle\Phi\,|H|\,\Phi\rangle}{\langle\Phi\mid\Phi\rangle} = \frac{\hbar^2}{2m}b^2 - e^2 b.$$

The minimum is at $\frac{dE(b)}{db} = 0$, that is at

$$b_{\min} = \frac{e^2 m}{\hbar^2}\ .$$

Inserting this value in $E(b)$ we find

$$E(b_{\min}) = -\left(\frac{e^2}{\hbar c}\right)^2 \frac{mc^2}{2}\ .$$

This variational result coincides with the exact ground state energy.

Problem 3: Statistical Mechanics I

(a) Consider a three-dimensional cubical box with a side length L. Recall that the single particle energies (assuming the wave function vanishes on the boundaries of the box) are given by

$$\epsilon_{n_x,n_y,n_z} = \frac{\hbar^2\pi^2}{2mL^2}(n_x^2 + n_y^2 + n_z^2) \ ,$$

where n_x, n_y, n_z three positive integer quantum numbers. Now consider N non-interacting fermions of spin 1/2 in such a three-dimensional box. Introducing a vector $\vec{n} = \{n_x, n_y, n_z\}$, each quantum state corresponds to a point in 'n-space' with energy $\epsilon_{\vec{n}} = \frac{\hbar^2\pi^2}{2mL^2}|\vec{n}|^2$. The number of states with energy less than ϵ_F is equal to the number of states that lie within a sphere of radius in the region of n-space where n_x, n_y, n_z are positive. In the ground state this number equals the number of fermions in the system:

$$N = 2 \times \frac{1}{8} \times \frac{4}{3}\pi n_F^3 \ ,$$

where the factor of two corresponds to two spin states, and the factor of 1/8 is because only 1/8 of the sphere lies in the region where all n's are positive. So, $n_F = (3N/\pi)^{1/3}$. The Fermi energy is then given by

$$\epsilon_F = \frac{\hbar^2\pi^2}{2mL^2}n_F^2 = \frac{\hbar^2\pi^2}{2mL^2}\left(\frac{3N}{\pi}\right)^{2/3} = \frac{\hbar^2}{2m}(3\pi^2\rho)^{2/3} \ .$$

Another solution (Y.A.): The single-particle density of states in 3D momentum space is a constant $g(\vec{k}) = 2V/(2\pi)^3$ (where the factor of 2 counts for spin degeneracy). To convert to density of states $g(\epsilon)$ (in energy ϵ), we use spherical coordinates in $\vec{k}$ and the dispersion relation $\epsilon = \hbar^2k^2/2m$:

$$g(\epsilon) = g(\vec{k})4\pi k^2 dk/d\epsilon = 2V/(2\pi)^3 4\pi k^2 m/(\hbar^2 k) = aV\sqrt{\epsilon} \ ,$$

where $a = \frac{1}{2\pi^2}(2m/\hbar^2)^{3/2}$. The Fermi energy is determined by

$$N = \int_0^{\epsilon_F} g(\epsilon)d\epsilon = \frac{2}{3}aV\epsilon_F^{3/2} \ ,$$

which leads to $\epsilon_F = \frac{\hbar^2}{2m}(3\pi^2\rho)^{2/3}$.

(b) The condition for the gas to be non-relativistic is

$$\epsilon_F = \frac{\hbar^2}{2m}(3\pi^2\rho)^{2/3} \lesssim mc^2$$

or

$$\rho \equiv \frac{N}{V} \lesssim \frac{2\sqrt{2}}{3\pi^2}\left(\frac{mc}{\hbar}\right)^3 \sim \left(\frac{mc}{\hbar}\right)^3 \ .$$

(c) For a non-relativistic gas, the total kinetic energy is

$$E_{\rm nr} \sim N\epsilon_F \sim N\left(\frac{N}{V}\right)^{2/3} \sim \frac{M^{5/3}}{R^2} \; .$$

Thus $E_{\rm nr} = C_1 \frac{M^{5/3}}{R^2}$, where C_1 is a constant.

(d) For a non-relativistic gas, the total energy of the system is given by

$$E_{tot} = C_1 \frac{M^{5/3}}{R^2} - C_2 \frac{M^2}{R} \; .$$

The system is stable if $dE_{tot}/dR = 0$ (i.e., the total energy is minimized);

$$\frac{dE_{tot}}{dR} = C_2 \frac{M^2}{R^2} - 2C_1 \frac{M^{5/3}}{R^3} = 0 \quad \text{at equilibrium} \; .$$

This condition holds for $R_0 = (2C_1/C_2)M^{-1/3}$. It's straightforward to check that $dE^2_{tot}/d^2R > 0$ for $R = R_0$. The system is thus stable with $R \propto M^{-1/3}$. This is a radius-mass relation of white dwarfs.

(e) Following a similar derivation as in (a), we find that for an ultra-relativistic gas (zero rest mass)

$$\epsilon_F \sim \rho^{1/3} \; .$$

Thus the total kinetic energy is

$$E_{\rm rel} \sim N\epsilon_F \sim N\left(\frac{N}{V}\right)^{1/3} \sim \frac{M^{4/3}}{R} \; .$$

(f) For a relativistic gas, the total energy of the system is

$$E_{tot} = C_1 \frac{M^{4/3}}{R} - C_2 \frac{M^2}{R} \; .$$

Setting $dE_{tot}/dR = 0$, we find that there exists a critical mass $M_c = (C_1/C_2)^{3/2}$. Notice from the above equation that gravity always wins (i.e., gravitational collapse) for $M > M_c$.

Problem 4: Statistical Mechanics II

(a) The photons do not interact and therefore thermal equilibrium cannot be reached without the presence of matter. Equilibration is achieved through the emission and absorption of photons by the walls of the cavity. As a result the number of photons is not fixed apriori.

The number of photons is determined by the equilibrium condition at constant T and V (i.e., the minimization of the free energy F): $(\partial F/\partial N)_{V,T} = 0$. Since $(\partial F/\partial N)_{V,T} = \mu$, we find that the chemical potential of the photon gas is

$$\mu = 0 \ .$$

Another argument: in deriving the Bose-Einstein partition function the chemical potential is introduced to satisfy the constraint that the average number of bosons is fixed. However, since the number of photons is not conserved we do not have to introduce a chemical potential and $\mu = 0$ in ,e.g., the expression for the average occupation. We can also think about the radiation field as a collection of harmonic oscillators. For an oscillator $\langle n \rangle = \frac{1}{e^{\beta\hbar\omega}-1}$ and again $\mu = 0$.

(b) For bosons with single-particle energies ϵ_r, $\ln Z = -\sum_r \ln[1 - e^{-(\epsilon_r - \mu)/kT}]$. Photons are bosons with $\mu = 0$ and $\epsilon = \hbar\omega = \hbar ck$. Converting the sum over r to an integral over ω, we have

$$F = -kT \ln Z = kT \int_0^\infty d\omega g(\omega) \ln(1 - e^{-\hbar\omega/kT}) \ ,$$

where $g(\omega)$ is the photon density of states.

To find $g(\omega)$, note that the density of states in 3D momentum space is $g(\vec{k}) = 2V/(2\pi)^3$, where the factor of 2 counts for the two possible polarizations of the photon. To convert to $g(\omega)$, we use spherical coordinates for $\vec{k}$ and $\omega = ck$:

$$g(\omega) = g(\vec{k}) 4\pi k^2 dk/d\omega = \frac{V\omega^2}{\pi^2 c^3} \equiv aV\omega^2 \ .$$

The free energy is then given by

$$F = kTaV \int_0^\infty d\omega \omega^2 \ln(1 - e^{-\hbar\omega/kT}) \ .$$

Substituting $x = \hbar\omega/kT$, we find

$$F = \hbar^{-3}(kT)^4 aV \int_0^\infty dx x^2 \ln(1 - e^{-x}) \ .$$

The integral is a negative numerical constant (can also be written as $-\frac{1}{3}\int_0^\infty dx \frac{x^3}{e^x-1})$, so we have

$$F \propto -VT^4 \ .$$

The (positive) entropy is then given by

$$S = -\left(\partial F/\partial T\right)_V \propto VT^3 \ .$$

(c) In an adiabatic process $S =$ const., so we have

$$VT^3 = V'T'^3 \ .$$

Using $V' = 2V$, we find $T' = \frac{T}{2^{1/3}}$.

(d) Using the expression for $\ln Z$ (as an integral over ω), we have

$$p = kT\left(\frac{\partial \ln Z}{\partial V}\right)_T = kTa\int_0^\infty d\omega\omega^2 \ln(1-e^{-\hbar\omega/kT}) \ .$$

Integrating by parts (the boundary term vanishes)

$$p = \frac{1}{3}a\hbar\int_0^\infty \frac{\omega^3}{e^{\hbar\omega/kT}-1} \ .$$

The r.h.s. can be written as follows

$$p = \frac{1}{3}\frac{1}{V}\int_0^\infty \hbar\omega\frac{aV\omega^2}{e^{\hbar\omega/kT}-1} = \frac{1}{3}\frac{1}{V}\int_0^\infty \hbar\omega g(\omega)n(\omega) \ ,$$

where $g(\omega)$ is the photon density of states and $n(\omega) = 1/(e^{\hbar\omega/kT}-1)$ is the occupation of a mode with frequency ω. The total average energy is given by $E = \int_0^\infty \hbar\omega g(\omega)n(\omega)$ and we obtain the equation of state

$$pV = \frac{1}{3}E \ .$$

(e) The pressure of non-interacting identical particles is given by

$$p = \sum_r \left(-\frac{\partial \epsilon_r}{\partial V}\right)\langle n_r\rangle \ ,$$

where ϵ_r is the energy of single-particle state r and $\langle n_r\rangle$ is the occupation of state r.

The dispersion of a zero rest mass particle is $\epsilon = c\hbar k$ where $\hbar\mathbf{k}$ is the momentum. Using the quantization condition of the momentum in a box of side L, $k_i = (2\pi/L)n_i$ (periodic b.c.), we find $\epsilon \propto V^{-1/3}$. It follows

$$-\frac{\partial\epsilon}{\partial V} = \frac{1}{3}\frac{\epsilon}{V} \ .$$

Using this relation in the above expression for p, we find

$$p = \frac{1}{3V}\sum_r \epsilon_r\langle n_r\rangle = \frac{1}{3}\frac{E}{V} \ ,$$

irrespective of the statistics.