QUALIFYING EXAMINATION, Part 1

Solutions

Problem 1: Mathematical Methods

(a) We expect two linearly independent solutions because it is a linear second order differential equation for $y(x)$.

(b) Plugging the series into the differential equation we find

$$\sum_{n=0}^{\infty} c_n \left[(n+s)(n+s-1)x^{n+s-1} - (n+s+1)x^{n+s}\right] = 0 \ ,$$

or

$$c_0 s(s-1)x^{s-1} + \sum_{n=0}^{\infty}(n+s+1)\left[(n+s)c_{n+1} - c_n\right]x^{n+s} = 0 \ ,$$

Demanding that the coefficient of the lowest power, x^{s-1} be zero, we find $s = 0, 1$.

For $s = 1$ we find the recursion relation

$$c_{n+1} = \frac{c_n}{n+1} \ ,$$

which means the series is

$$y = c_0 \sum_{0}^{\infty} \frac{x^{n+1}}{n!} = c_0 x e^x \ .$$

(c)

Setting

$$y = xe^x F(x)$$

and solving for F, we find

$$xF'' + (x+2)F' = 0 \ ,$$

which means

$$\begin{aligned}
\frac{F''}{F'} &= -\left(\frac{2}{x} + 1\right) \\
\ln F' &= -2\ln x - x \\
F' &= c\frac{e^{-x}}{x^2} \\
y &= cxe^x \int^x \frac{e^{-x'}}{x'^2} dx' \ .
\end{aligned}$$

Expanding the integrand in powers up to x'^3 and integrating, we find

$$y(x) = ce^x \left(1 + x \ln x - \frac{x^2}{2} + \frac{x^3}{12} + \mathcal{O}(x^4)\right)$$

for small x.

Problem 2: Classical Mechanics

(a) The kinetic and potential energies are given by

$$T = \frac{1}{2} M \dot{x}^2 \,, \qquad V = 2 \frac{1}{2} k x^2 = k x^2$$

so the Lagrangian is

$$\mathcal{L} = \frac{1}{2} M \dot{x}^2 - k x^2 \,.$$

Euler-Lagrange equation of motion is

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = M \ddot{x} + 2kx = 0 \,.$$

Thus $\ddot{x} + \omega^2 x = 0$ with $\omega = \sqrt{2k/M}$.

(b) When the mass is at $(x.y)$ the extension of each spring is $\sqrt{(L + S \pm x)^2 + y^2} - L$. Thus the potential energy $V_\pm$ of each spring is

$$\begin{aligned} V_\pm &= \frac{1}{2} k \left[\sqrt{(L + S \pm x)^2 + y^2} - L \right]^2 \\ &= \frac{1}{2} k \left[(L + S \pm x)^2 + y^2 + L^2 - 2L \sqrt{(L + S \pm x)^2 + y^2} \right] \,. \end{aligned}$$

For $x, y \ll L + S \equiv \lambda$, we expand to second order in x, y. Dropping terms that are constants and linear in x, y (see hint), we find

$$\begin{aligned} V_\pm(x,y) - V_\pm(0,0) &= \frac{1}{2} k \left[x^2 + y^2 - 2L\lambda \sqrt{1 \pm \frac{2x}{\lambda} + \left(\frac{x}{\lambda}\right)^2 + \left(\frac{y}{\lambda}\right)^2} \right] \\ &= \frac{1}{2} k \left[x^2 + y^2 - 2L\lambda \left\{ \frac{1}{2} \left(\frac{x}{\lambda}\right)^2 + \frac{1}{2} \left(\frac{y}{\lambda}\right)^2 - \frac{1}{8} \left(\frac{2x}{\lambda}\right)^2 \right\} \right] \\ &= \frac{1}{2} k \left(x^2 + y^2 - L \frac{y^2}{\lambda} \right) = \frac{1}{2} k \left(x^2 + \frac{S}{L+S} y^2 \right) \,. \end{aligned}$$

Adding V_+ to V_- gives

$$V(x,y) - V(0,0) = k \left(x^2 + \frac{S}{L+S} y^2 \right) \,.$$

The x and y directions are thus uncoupled and correspond to the normal modes of the system. The oscillation frequency in x is the same as before

$$\omega_x = \sqrt{2k/M} \,,$$

while the oscillation frequency for the new coordinate y is

$$\omega_y = \sqrt{\frac{S}{L+S}\frac{2k}{M}}\ .$$

(c) To keep the mass rotating about the x-axis, the centripetal force must be equal to $M\omega^2\rho$. This force is equal to the projection of the spring forces perpendicular to x, i.e., $kS\tan\theta$ (to first order in S), where $\tan\theta = \rho/(L+S)$. We find

$$M\omega^2\rho = 2kS\rho/(L+S), \tag{1}$$

so $\omega = \sqrt{\dfrac{S}{L+S}\dfrac{2k}{M}}$.

In part (b) we found a normal mode in the y direction with frequency ω_y. By symmetry, there is another normal mode in the z direction with the same frequency $\omega_z = \omega_y = \omega$. the oscillation frequency in the z direction should be the same as in the y direction. Any superposition of these two degenerate modes is a solution. Taking the same amplitude but a phase difference of $\pi/2$ between the two modes, we obtain the above circular solution.

Problem 3: Electromagnetism I

(a) General solution:

Laplace equation holds inside the box

$$\nabla^2\Phi(x,y,z) = 0 \ .$$

Using separation of variables in cartesian coordinates, we assume $\Phi(x,y,z) = X(x)Y(y)Z(z)$, and Laplace equation becomes

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0 \ .$$

Each term must be a constant, leading to

$$\begin{aligned} \frac{1}{X}\frac{d^2X}{dx^2} &= -\alpha^2 \\ \frac{1}{Y}\frac{d^2Y}{dy^2} &= -\beta^2 \\ \frac{1}{Z}\frac{d^2Z}{dz^2} &= \gamma^2 \end{aligned}$$

with $\alpha^2 + \beta^2 = \gamma^2$. The general solution is given by

$$\begin{aligned} X &= A\cos(\alpha x) + B\sin(\alpha x) \\ Y &= C\cos(\beta y) + D\sin(\beta y) \\ Z &= E\exp(\gamma z) + F\exp(-\gamma z) \ . \end{aligned}$$

Next we employ the boundary conditions.

(b) Vertical sides:

$$\left.\frac{\partial\Phi}{\partial x}\right|_{x=0,a} = 0 = \left.\frac{\partial\Phi}{\partial y}\right|_{y=0,b} \ .$$

It follows that:

$$B = D = 0 \ ,$$

and

$$\begin{aligned} \alpha = \alpha_m &= \frac{m\pi}{a} \ , \ m \in \mathbb{N} \\ \beta = \beta_n &= \frac{n\pi}{b} \ , \ n \in \mathbb{N} \\ \gamma = \gamma_{mn} &= \pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \ . \end{aligned}$$

(c) Bottom face:

$$\Phi(x, y, 0) = 0 \ .$$

This requires

$$F = -E \ .$$

Putting (b) and (c) together, we have

$$\Phi_{mn}(x, y, z) = A_{mn} \cos(\alpha_m x) \cos(\beta_n y) \sinh(\gamma_{mn} z) \ ,$$

and the general solution becomes:

$$\Phi(x, y, z) = \sum_{m,n=1}^{\infty} \Phi_{mn}(x, y, z) \tag{2}$$

(d) Top face:

For $z = c$ we use Ohm's law to account for the wire, and the contact point is modeled by a delta function

$$\left. \frac{\partial \Phi}{\partial z} \right|_{z=c} = -E_z = -\frac{I}{\sigma} \delta\left(x - \frac{a}{2}\right) \delta\left(y - \frac{b}{2}\right) \ .$$

Putting this together with the general solution derived above gives:

$$\sum_{m,n=1}^{\infty} A_{mn} \cos(\alpha_m x) \cos(\beta_n y) \gamma_{mn} \cosh(\gamma_{mn} c) = \frac{I}{\sigma} \delta\left(x - \frac{a}{2}\right) \delta\left(y - \frac{b}{2}\right) \ .$$

This relation describes the expansion of the function on the r.h.s. in terms of a complete and orthogonal set of functions $\cos(\alpha_m x) \cos(\beta_n y)$. The coefficients $A_{m,n}$ can be calculated as usual by multiplying both sides by $\cos(\alpha_{m'} x) \cos(\beta_{n'} y)$ and integrating over the top surface using the orthonormality relations.

The final result (not requested in the problem) is

$$\begin{aligned} A_{mn} &= \frac{4I}{ab\sigma\gamma_{mn} \cosh(\gamma_{mn} c)} \int_0^a \int_0^b \delta\left(x - \frac{a}{2}\right) \delta\left(y - \frac{b}{2}\right) \cos(\alpha_m x) \cos(\beta_n y) dx dy \\ &= \frac{4I}{ab\sigma\gamma_{mn} \cosh(\gamma_{mn} c)} \cos\left(\frac{m\pi}{2}\right) \cos\left(\frac{n\pi}{2}\right) \ . \end{aligned}$$

We note that only even values of n and m contribute to the general solution.

Problem 4: Electromagnetism II

This solution uses rationalized Gaussian units.

(a) The surface charge density on the sphere is

$$\sigma = \frac{Q}{4\pi a^2} \ .$$

The surface current density is $\vec{K} = \sigma\vec{v}$, where $\vec{v} = \vec{r} \times (\omega\hat{z}) = \omega a \sin\theta\hat{\phi}$ is the velocity at position $\vec{r}$ on the sphere. Thus

$$\vec{K} = \frac{Q\omega}{4\pi a} \sin\theta\hat{\phi} \ .$$

(b) Applying Gauss' law to a sphere of radius r, we find

$$\vec{E}(r < a) = 0 \ , \quad \vec{E}(r > a) = \frac{Q}{4\pi r^2}\hat{r} \ .$$

(c) The magnetic moment can be calculated by breaking up the current distribution into a stack of infinitesimal rings or radius $a \sin\theta$ carrying current

$$dI = (\text{surface current}) \cdot (\text{arc length}) = \left(\frac{Q}{4\pi a^2}\omega a \sin\theta\right) \cdot a d\theta,$$

and integrating. Thus

$$\vec{\mu} = \frac{\hat{z}}{c}\int dI(\theta)\,\text{Area}(\theta) = \frac{Qa^2\omega}{4c}\hat{z}\int_0^\pi d\theta \sin^3\theta = \frac{Q\omega a^2}{3c}\hat{z}.$$

(d) $\vec{\nabla} \times \vec{B} = 0$ for $r \neq a$, so can introduce a magnetic potential Φ_M with $\vec{B} = -\vec{\nabla}\Phi_M$ for $r \neq a$. Φ_M satisfies Laplace equation for $r \neq a$. We are given that only the $\ell = 1$ terms (of the general solution) contribute to the field

$$\Phi_M = \begin{cases} A_i r \cos\theta & (r < a) \\ A_o \cos\theta / r^2 & (r > a) \end{cases}$$

To determine A_i and A_o we use the following boundary conditions on $\vec{B}$ at the surface of the sphere:

(i) The component of $\vec{B}$ perpendicular to the surface, i.e., the radial component B_r, is continuous across $r = a$. Using $B_r = -\partial\Phi_M/\partial r$ we find

$$A_i = -\frac{2A_o}{a^3} \ .$$

(ii) The component of $\vec{B}$ parallel to the surface and perpendicular to the surface current density, i.e., B_θ, has a jump of K/c across the surface of the sphere. Using $B_\theta = -\frac{1}{r}\frac{\partial \Phi_M}{\partial \theta}$, we find

$$\frac{A_o \sin\theta}{a^3} - A_i \sin\theta = \frac{Q\omega}{4\pi ac}\sin\theta \ ,$$

or

$$\frac{A_o}{a^3} - A_i = \frac{Q\omega}{4\pi ac} \ .$$

Solving the two equations for A_i and A_o we find

$$A_i = -\frac{Q\omega}{2\pi ac} \ , \quad A_o = \frac{Q\omega a^2}{12\pi c} \ .$$

Using $\vec{B} = -\vec{\nabla}\Phi_M$, we obtain

$$\vec{B} = \begin{cases} \frac{Q\omega}{6\pi ac}\hat{z} & (r < a), \\ \frac{Q\omega}{12\pi c}\frac{a^2}{r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right) & (r > a). \end{cases}$$

Using $\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$ and the expression for $\vec{\mu}$ in part (c), we can write the field outside the sphere $(r > a)$ as

$$\vec{B} = \frac{\mu}{4\pi}\frac{1}{r^3}\left(3\cos\theta\,\hat{r} - \hat{z}\right) = \frac{1}{4\pi}\frac{1}{r^3}\left[3(\vec{\mu}\cdot\hat{r})\hat{r} - \vec{\mu}\right] \ .$$

which is the field of a magnetic dipole $\vec{\mu}$.

(e) The EMF through a circle of constant θ and $r = a$ is given by $2\pi a \sin\theta E_\phi(r = a, \theta)$, which by Faraday's law is the rate of change of the magnetic flux through the surface enclosed by the loop. Using the expression in part (d) for the magnetic field inside the sphere, this flux is $\frac{Q\omega}{6\pi ac}\pi(a^2\sin^2\theta)$. Thus

$$E_\phi(r = a, \theta) = -\frac{Q\dot{\omega}}{12\pi c^2}\sin\theta \ .$$