QUALIFYING EXAMINATION, Part 1

Solutions

Problem 1: Mathematical Methods

(a) For \( r > 0 \) we find

\[
\nabla^2 \left( \frac{1}{r} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \left( \frac{1}{r} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \left( -\frac{1}{r^2} \right) = 0
\]

However for \( r = 0 \) we get \( \frac{1}{r^2} \) because of the \( \frac{1}{r^2} \) factor in front. To calculate the limit \( r \to 0 \), we integrate \( \nabla^2 \left( \frac{1}{r} \right) \) over a small sphere of radius \( a \) surrounding the origin and use Gauss’ theorem (converting a volume integral to a surface integral)

\[
\int_{r \leq a} d^3 \vec{r} \nabla^2 \left( \frac{1}{r} \right) = \int_{r \leq a} d^3 \vec{r} \nabla \cdot \nabla \left( \frac{1}{r} \right)
= \int_{r=a} \nabla \left( \frac{1}{r} \right) \cdot d\vec{S} = \int_{r=a} \left( -\frac{1}{a^2} \right) \hat{e}_r \cdot d\vec{S} = -4\pi \cdot
\]

It follows that

\[
\nabla^2 \left( \frac{1}{r} \right) = -4\pi \delta^3(\vec{r}).
\]

Note: \( \frac{1}{r} \) is the solution to Poisson’s equation with a point charge at the origin with charge \( 4\pi \epsilon_0 \).

(b) \[
\Gamma(1) = \int_0^\infty t^0 e^{-t} dt = - e^{-t} \bigg|_0^\infty = 1 .
\]

(c) \[
\Gamma(x + 1) = \int_0^\infty t^x e^{-t} dt = -t^x e^{-t} \bigg|_0^\infty + \int_0^\infty t^{x-1} e^{-t} dt = 0 + x \Gamma(x) ,
\]

where the surface term vanishes at \( t = 0 \) if \( x \geq 0 \).

It follows that \( \Gamma(n + 1) = n \Gamma(n) \) which, given \( \Gamma(1) = 1 \), means

\[
\Gamma(n + 1) = n! .
\]
\( \ln n! = \ln [n(n-1)(n-2) \ldots 1] = \sum_{m=1}^{n} \ln m \simeq \int_{1}^{n} \ln x dx = n \ln n - n + 1 \),

where we can drop the 1 compared to \( n \). Exponentiating, we find

\( n! \approx e^{n \ln n - n} = n^n e^{-n} \).

(e) We have

\( n! = \Gamma(n+1) = \int_{0}^{\infty} t^n e^{-t} dt \).

We rescale \( t = nz \) to obtain

\[ n! = n^{n+1} \int_{0}^{\infty} z^n e^{-nz} dz = n^{n+1} \int_{0}^{\infty} e^{-nz + n \ln z} dz \equiv n^{n+1} \int_{0}^{\infty} e^{-nf(z)} dz, \]

where

\( f(z) = z - \ln z \).

The saddle point \( z = z^* \) is determined by the condition \( f'(z) = z - 1/z = 0 \), giving \( z^* = 1 \). We expand \( f(z) \) to second order around \( z^* = 1 \) to find

\[ \Gamma(n+1) = n^{n+1} \int_{0}^{\infty} e^{-nf(z)} dz \simeq n^{n+1} e^{-nf(z^*)} \int_{0}^{\infty} e^{-n f''(z^*)(z-z^*)^2} dz. \]

Using \( f''(z^*) = \frac{1}{z^2} \bigg|_{z^*} = 1 \), we obtain

\[ n! = n^n e^{-n} \sqrt{2\pi n} \ (1 + \text{corrections}), \]

where we have changed variables to \( w = z - z^* \) and extended the limits to \( w = \pm \infty \) (the integrand is negligible far away from \( z^* \)). We can then use the familiar result for a Gaussian integral

\[ \int_{-\infty}^{\infty} e^{-aw^2} dw = \sqrt{\frac{\pi}{a}}. \]

The initial rescaling was done to get a large factor \( n \) in the exponent in front of \( f(z) \) (to justify the saddle-point approximation). It is also possible to carry out the saddle-point approximation directly for \( f(t) = n \ln t - t \), in which case the saddle-point condition \( f'(t^*) = 0 \) gives \( t^* = n \).
Problem 2: Classical Mechanics

(a) The first Lagrangian is invariant under translations in the x direction \( x \rightarrow x + a \) and therefore \( p_x \) (the generator of translations in the x direction) is conserved. It does not depend on time explicitly so \( \partial L/\partial t = -dH/dt = 0 \) and the Hamiltonian \( H = E \) is conserved. The Lagrangian is not invariant under translations in y or under rotations, and therefore \( p_y \) and \( L_z \) are not conserved.

The second Lagrangian is not invariant under translations (in x or y) and therefore \( p_x \) and \( p_y \) are not conserved. It is rotationally invariant and explicitly time independent, and therefore \( L_z \) and \( E \) are conserved.

The third Lagrangian is invariant under translations in x but not under translations in y or under rotations. Therefore \( p_x \) is conserved but \( p_y \) and \( L_z \) are not conserved. Because of the explicit time dependence in \( L \), \( E \) is not conserved.

(b) The potential is time independent, and hence the total energy is conserved:

\[
E = E_0 = \frac{1}{2}mv_0^2 = \text{const.}
\]

The potential is central and provides no torque, hence the angular momentum around the scattering center is conserved

\[
\vec{L} = \vec{L}_0 = mv_0b\hat{z} = \text{const.}
\]

\( L_x = L_y = 0 \) guaranteeing that the motion is in the x-y plane.

(c) Using the effective potential for the reduced 1D motion in the radial coordinate \( r \), the energy can be written as

\[
E = \frac{1}{2}mr^2 + U_{\text{eff}}(r) = \frac{1}{2}mr^2 + \frac{L^2}{2mr^2} - \frac{k}{r^2} = \frac{1}{2}mv_0^2.
\]

At the distance \( r = r_{\text{min}} \) of closest approach \( \dot{r} = 0 \), and we have

\[
\frac{(mv_0b)^2}{2mr_{\text{min}}^2} - \frac{k}{r_{\text{min}}^2} = \frac{1}{2}mv_0^2.
\]

Therefore

\[
r_{\text{min}} = \sqrt{\frac{mv_0^2b^2/2 - k}{mv_0^2/2}} = \sqrt{b^2 - \frac{2k}{mv_0^2}}.
\]

Since \( k > 0 \), \( r_{\text{min}} < b \). Qualitatively this is expected since the potential is attractive.
(d) The horizontal coordinate of the point particle is given by \( x + X \), while its vertical coordinate is \( y = x \tan \alpha \). The kinetic energy \( T \) of the system is then

\[
T = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} m (\dot{X} + \dot{x})^2 + \frac{1}{2} m y^2 = \frac{1}{2} (M + m) \dot{X}^2 + m \dot{X} \dot{x} + \frac{1}{2} m (1 + \tan^2 \alpha) \dot{x}^2
\]

The potential energy of the particle is

\[
V = mg y = mg x \tan \alpha.
\]

The Lagrangian of the system is therefore:

\[
\mathcal{L} = T - V = \frac{1}{2} (M + m) \dot{X}^2 + m \dot{X} \dot{x} + \frac{1}{2} m (1 + \tan^2 \alpha) \dot{x}^2 - mg x \tan \alpha.
\]

The equations of motion are:

\[
dt \left( \frac{\partial L}{\partial \dot{X}} \right) - \frac{\partial L}{\partial X} = 0 \rightarrow (M + m) \ddot{X} + m \ddot{x} = 0.
\]

\[
dt \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \rightarrow m \ddot{X} + m (1 + \tan^2 \alpha) \ddot{x} + mg \tan \alpha = 0.
\]

From the first equation we get

\[
\ddot{X} = - \frac{m}{M + m} \ddot{x}.
\]

Substituting in the second equation, we find

\[
\ddot{x} = - \frac{(M + m) g \sin \alpha \cos \alpha}{M + m \sin^2 \alpha}.
\]

We then have

\[
\ddot{X} = \frac{mg \sin \alpha \cos \alpha}{M + m \sin^2 \alpha}.
\]
Problem 3: Electromagnetism I

(a) Using Coulomb’s law

\[ V(r, \theta) = \frac{q}{4\pi\epsilon_0} \left( \frac{1}{r_+} + \frac{1}{r_-} - \frac{2}{r} \right) \]

where (applying the law of cosines)

\[ r_\pm = \left( r^2 + a^2 \mp 2ra \cos \theta \right)^{1/2}. \]

(b) Expanding \(1/r_\pm\) to order \(a^2/r^2\) (all higher order terms will vanish in the limit), we have

\[ \frac{1}{r} \approx \frac{1}{r} \left( 1 - \frac{a^2}{2r^2} \pm \frac{a}{r} \cos \theta + \frac{3a^2}{2r^2} \cos^2 \theta \right). \]

Substituting into the potential, we get

\[ V(r, \theta) = \frac{Q}{4\pi\epsilon_0 r^3} \left( 3 \cos^2 \theta - 1 \right) = \frac{Q}{2\pi\epsilon_0 r^3} P_2(\cos \theta), \]

where we have used the expression for the second Legendre polynomial \(P_2\).

(c) Since the conducting sphere is grounded \(V(r = R) = 0\). We also know that \(V(r \to \infty) = 0\). The uniqueness of the solution of Laplace’s equation with boundary conditions gives us \(V(r) = 0\) outside the sphere.

(d) Inside the sphere, we use the general solution of Laplace’s equation

\[ V_{\text{in}} = \sum \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta). \]

The potential is continuous across \(r = R\). Using the result in (c), we have

\[ V_{\text{in}}(R, \theta) = 0. \]

Since this is true for all angles \(\theta\), the sum must vanish term by term. This gives

\[ -A_l R^l = \frac{B_l}{R^{l+1}}. \]

In the limit \(r \to 0\) \((r \ll R)\), we match the solution to the result from part. This gives us

\[ B_l \neq 2, \quad B_2 = \frac{Q}{2\pi\epsilon_0}. \]

Using the first set of relations between \(A_l\) and \(B_l\), we find
Thus the potential inside is given by

\[ V_{\text{in}}(r, \theta) = \frac{Q}{2\pi \epsilon_0 r^3} \left( 1 - \frac{r^3}{R^5} \right) P_2(\cos \theta). \]
Problem 4: Electromagnetism II

(a) The voltage per turn has to balance the induced electro-motive force (emf)

\[ V = -\varepsilon = \frac{d\Phi}{dt} = \pi a^2 \frac{dB}{dt} \]

(b) Using a rectangular Amperian loop outside the solenoid, one can show that the field outside (parallel to the cylinder axis) is independent of the radial distance \( r \). Since the field vanishes at \( r \rightarrow \infty \), it must be zero everywhere outside the cylinder.

(c) From symmetry considerations \( \vec{B} \) is along the cylinder’s axis. Using a rectangular Amperian loop for the field \( H \) with one side inside the solenoid and another side outside, gives

\[ H = nI \quad \text{or} \quad I = \frac{B(t)}{\mu n} . \]

(d) We use a circular loop \( C \) around the cylinder with radius \( r \). By symmetry the magnitude of \( \vec{A} \) is constant along the loop and is tangential along the same direction as the current through the curled wire. Using Stokes theorem with \( \nabla \times \vec{A} = \vec{B} \) we have

\[ \oint_C \vec{A} \cdot d\vec{l} = \int_S \vec{B} \cdot d\vec{S} \equiv \Phi , \]

where \( \Phi \) is the magnetic flux through the loop.

We find for \( r > a \)

\[ 2\pi r A(r, t) = B(t)\pi a^2 , \]

\[ A(r, t) = \frac{a^2}{2r} B(t) , \]

and for \( r < a \)

\[ 2\pi r A(r, t) = \pi r^2 B(t) , \]

\[ A(r, t) = \frac{r}{2} B(t) . \]

(e) The field is uniform inside the torus. Since \( \nabla \cdot \vec{B} = 0 \), the flux of \( B \) is constant around the torus. Since the cross section of the torus is the same everywhere, \( B \) must be a constant inside the torus. Since the gap \( h \) is small, \( B \) in the gap will be near homogeneous (negligible fringe fields) and will have the same value as inside the torus.

To find the value of \( B \), we use a circular Amperean loop of radius \( Z \) around the torus. We have

\[ \oint_Z \vec{H} d\vec{l} = NI \]
or
\[
\frac{B}{\mu} (2\pi Z - h) + \frac{B}{\mu_0} h = NI .
\]

This gives
\[
B = \frac{\mu \mu_0 NI}{\mu_0 (2\pi Z - h) + \mu h} \approx \frac{\mu \mu_0 NI}{\mu_0 2\pi Z + \mu h} .
\]

The current \( I \) in the wire is \( I = \frac{V}{\pi} \), giving the final answer
\[
B = \frac{\mu \mu_0 NV}{R (2\mu_0 \pi Z + \mu h)} .
\]