QUALIFYING EXAMINATION, Part 1

Solutions

Problem 1: Mathematical Methods

(a) For $r > 0$ we find

$$\nabla^2\left(\frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\left(-\frac{1}{r^2}\right) = 0$$

However for $r = 0$ we get $\frac{0}{0}$ because of the $\frac{1}{r^2}$ factor in front. To calculate the limit $r \to 0$, we integrate $\nabla^2\left(\frac{1}{r}\right)$ over a small sphere of radius a surrounding the origin and use Gauss' theorem (converting a volume integral to a surface integral)

$$\begin{aligned}\int_{r\leq a} d^3\vec{r}\,\nabla^2\left(\frac{1}{r}\right) &= \int_{r\leq a} d^3\vec{r}\,\nabla\cdot\nabla\left(\frac{1}{r}\right)\\ &= \int_{r=a}\nabla\left(\frac{1}{r}\right)\cdot d\vec{S} = \int_{r=a}\left(-\frac{1}{a^2}\right)\hat{\mathbf{e}}_r\cdot d\vec{S} = -4\pi\ .\end{aligned}$$

It follows that

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\vec{r})\ .$$

Note: $\frac{1}{r}$ is the solution to Poisson's equation with a point charge at the origin with charge $4\pi\epsilon_0$.

(b)

$$\Gamma(1) = \int_0^\infty t^0 e^{-t}dt = -\left. e^{-t}\right|_0^\infty = 1\ .$$

(c)

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}dt = -t^x e^{-t}\Big|_0^\infty + \int_0^\infty x\, t^{x-1}e^{-t}dt = 0 + x\Gamma(x)\ ,$$

where the surface term vanishes at $t = 0$ if $x \geq 0$.

It follows that $\Gamma(n+1) = n\Gamma(n)$ which, given $\Gamma(1) = 1$, means

$$\Gamma(n+1) = n!\ .$$

(d)

$$\ln n! = \ln\left[n(n-1)(n-2)\ldots 1\right] = \sum_{m=1}^{n} \ln m \simeq \int_1^n \ln x dx = n\ln n - n + 1 \ ,$$

where we can drop the 1 compared to n. Exponentiating, we find

$$n! \approx e^{n\ln n - n} = n^n e^{-n} \ .$$

(e) We have

$$n! = \Gamma(n+1) = \int_0^\infty t^n e^{-t} dt \ .$$

We rescale $t = nz$ to obtain

$$n! = n^{n+1}\int_0^\infty z^n e^{-nz} dz = n^{n+1}\int_0^\infty e^{-nz+n\ln z} dz \equiv n^{n+1}\int_0^\infty e^{-nf(z)} dz \ ,$$

where

$$f(z) = z - \ln z \ .$$

The saddle point $z = z^*$ is determined by the condition $f'(z) = z - 1/z = 0$, giving $z^* = 1$. We expand $f(z)$ to second order around $z^* = 1$ to find

$$\Gamma(n+1) = n^{n+1}\int_0^\infty e^{-nf(z)} dz \simeq n^{n+1} e^{-nf(z^*)} \int_0^\infty e^{-n\frac{1}{2!}f"(z^*)(z-z^*)^2} dz \ .$$

Using $f''(z^*) = \left.\frac{1}{z^2}\right|_{z^*} = 1$, we obtain

$$n! = n^n e^{-n}\sqrt{2\pi n}\,\left(1 + \text{corrections}\right) \ ,$$

where we have changed variables to $w = z - z^*$ and extended the limits to $w = \pm\infty$ (the integrand is negligible far away from z^*). We can then use the familiar result for a Gaussian integral

$$\int_{-\infty}^{\infty} e^{-\alpha w^2} dw = \sqrt{\frac{\pi}{\alpha}} \ .$$

The initial rescaling was done to get a large factor n in the exponent in front of $f(z)$ (to justify the saddle-point approximation). It is also possible to carry out the saddle-point approximation directly for $f(t) = n\ln t - t$, in which case the saddle-point condition $f'(t^*) = 0$ gives $t^* = n$.

Problem 2: Classical Mechanics

(a) The first Lagrangian is invariant under translations in the x direction $x \to x + a$ and therefore p_x (the generator of translations in the x direction) is conserved. It does not depend on time explicitly so $\partial\mathcal{L}/\partial t = -dH/dt = 0$ and the Hamiltonian $H = E$ is conserved. The Lagrangian is not invariant under translations in y or under rotations, and therefore p_y and L_z are not conserved.

The second Lagrangian is not invariant under translations (in x or y) and therefore p_x and p_y are not conserved. It is rotationally invariant and explicitly time independent, and therefore L_z and E are conserved.

The third Lagrangian is invariant under translations in x but not under translations in y or under rotations. Therefore p_x is conserved but p_y and L_z are not conserved. Because of the explicit time dependence in $\mathcal{L}$, E is not conserved.

(b) The potential is time independent, and hence the total energy is conserved:

$$E = E_0 = \frac{1}{2}mv_0^2 = \text{const.}$$

The potential is central and provides no torque, hence the angular momentum around the scattering center is conserved

$$\vec{L} = \vec{L}_0 = mv_0 b\hat{z} = \text{const.}$$

.

$L_x = L_y = 0$ guaranteeing that the motion is in the x-y plane.

(c) Using the effective potential for the reduced 1D motion in the radial coordinate r, the energy can be written as

$$E = \frac{1}{2}m\dot{r}^2 + U_{\text{eff}}(r) = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{k}{r^2} = \frac{1}{2}mv_0^2 \ .$$

At the distance $r = r_{\text{min}}$ of closest approach $\dot{r} = 0$, and we have

$$\frac{(mv_0 b)^2}{2mr_{\text{min}}^2} - \frac{k}{r_{\text{min}}^2} = \frac{1}{2}mv_0^2 \ .$$

Therefore

$$r_{min} = \sqrt{\frac{mv_0^2 b^2/2 - k}{mv_0^2/2}} = \sqrt{b^2 - \frac{2k}{mv_0^2}} \ .$$

Since $k > 0$, $r_{\text{min}} < b$. Qualitatively this is expected since the potential is attractive.

(d) The horizontal coordinate of the point particle is given by $x + X$, while its vertical coordinate is $y = x \tan\alpha$. The kinetic energy T of the system is then

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{X} + \dot{x})^2 + \frac{1}{2}m\dot{y}^2 = \frac{1}{2}(M+m)\dot{X}^2 + m\dot{X}\dot{x} + \frac{1}{2}m(1 + \tan^2\alpha)\dot{x}^2$$

The potential energy of the particle is

$$V = mgy = mgx\tan\alpha \ .$$

The Lagrangian of the system is therefore:

$$\mathcal{L} = T - V = \frac{1}{2}(M+m)\dot{X}^2 + m\dot{X}\dot{x} + \frac{1}{2}m(1+\tan^2\alpha)\dot{x}^2 - mgx\tan\alpha \ .$$

The equations of motion are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) - \frac{\partial L}{\partial X} = 0 \rightarrow (M+m)\ddot{X} + m\ddot{x} = 0 \ .$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \rightarrow m\ddot{X} + m(1+\tan^2\alpha)\ddot{x} + mg\tan\alpha = 0 \ .$$

From the first equation we get

$$\ddot{X} = -\frac{m}{M+m}\ddot{x} \ .$$

Substituting in the second equation, we find

$$\ddot{x} = -\frac{(M+m)g\sin\alpha\cos\alpha}{M + m\sin^2\alpha} \ .$$

We then have

$$\ddot{X} = \frac{mg\sin\alpha\cos\alpha}{M+m\sin^2\alpha} \ .$$

Problem 3: Electromagnetism I

(a) Using Coulomb's law

$$V(r,\theta) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_+} + \frac{1}{r_-} - \frac{2}{r}\right)$$

where (applying the law of cosines)

$$r_\pm = \left(r^2 + a^2 \mp 2ra\cos\theta\right)^{1/2} \ .$$

(b) Expanding $1/r_\pm$ to order a^2/r^2 (all higher order terms will vanish in the limit), we have

$$\frac{1}{r_\pm} \approx \frac{1}{r}\left(1 - \frac{1}{2}\frac{a^2}{r^2} \pm \frac{a}{r}\cos\theta + \frac{3}{2}\frac{a^2}{r^2}\cos^2\theta\right) \ .$$

Substituting into the potential, we get

$$V(r,\theta) = \frac{Q}{4\pi\epsilon_0 r^3}\left(3\cos^2\theta - 1\right) = \frac{Q}{2\pi\epsilon_0 r^3}P_2(\cos\theta) \, ,$$

where we have used the expression for the second Legendre polynomial P_2.

(c) Since the conducting sphere is grounded $V(r = R) = 0$. We also know that $V(r \to \infty) = 0$. The uniqueness of the solution of Laplace's equation with boundary conditions gives us $V(r) = 0$ outside the sphere.

(d) Inside the sphere, we use the general solution of Laplace's equation

$$V_{\text{in}} = \sum\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) \ .$$

The potential is continuous across $r = R$. Using the result in (c), we have

$$V_{\text{in}}(R,\theta) = 0 \ .$$

Since this is true for all angles θ, the sum must vanish term by term. This gives

$$-A_l R^l = \frac{B_l}{R^{l+1}} \ .$$

In the limit $r \to 0$ ($r \ll R$), we match the solution to the result from part. This gives us

$$B_{l\neq 2} = 0 \ , \quad B_2 = \frac{Q}{2\pi\epsilon_0} \ .$$

Using the first set of relations between A_l and B_l, we find

$$A_{l\neq 2} = 0\ , \quad A_2 = -\frac{B_2}{R^5}\ .$$

Thus the potential inside is given by

$$V_{\rm in}(r,\theta) = \frac{Q}{2\pi\epsilon_0 r^3}\left(1 - \frac{r^5}{R^5}\right) P_2(\cos\theta)\,.$$

Problem 4: Electromagnetism II

(a) The voltage per turn has to balance the induced electro-motive force (emf)

$$V = -\varepsilon = \frac{d\Phi}{dt} = \pi a^2 \frac{dB}{dt}$$

(b) Using a rectangular Amperian loop outside the solenoid, one can show that the field outside (parallel to the cylinder axis) is independent of the radial distance r. Since the field vanishes at $r \to \infty$, it must be zero everywhere outside the cylinder.

(c) From symmetry considerations $\vec{B}$ is along the cylinder's axis. Using a rectangular Amperian loop for the field H with one side inside the solenoid and another side outside, gives

$$H = nI \quad \text{or} \quad I = \frac{B(t)}{\mu n} \ .$$

(d) We use a circular loop C around the cylinder with radius r. By symmetry the magnitude of $\vec{A}$ is constant along the loop and is tangential along the same direction as the current through the curled wire. Using Stokes theorem with $\nabla \times \vec{A} = \vec{B}$ we have

$$\oint_C \vec{A} \cdot d\vec{l} = \int_S \vec{B} \cdot d\vec{S} \equiv \Phi \ ,$$

where Φ is the magnetic flux through the loop.

We find for $r > a$

$$2\pi r A(r,t) = B(t)\pi a^2 \ ,$$
$$A(r,t) = \frac{a^2}{2r} B(t) \ ,$$

and for $r < a$

$$2\pi r A(r,t) = \pi r^2 B(t) \ ,$$
$$A(r,t) = \frac{r}{2} B(t) \ .$$

(e) The field is uniform inside the torus. Since $\nabla \cdot \vec{B} = 0$, the flux of B is constant around the torus. Since the cross section of the torus is the same everywhere, B must be a constant inside the torus. Since the gap h is small, B in the gap will be near homogeneous (negligible fringe fields) and will have the same value as inside the torus.

To find the value of B, we use a circular Ampreian loop of radius Z around the torus. We have

$$\oint_Z \vec{H} d\vec{l} = NI$$

or

$$\frac{B}{\mu}(2\pi Z - h) + \frac{B}{\mu_0}h = NI \ .$$

This gives

$$\begin{aligned} B &= \frac{\mu\mu_0 NI}{\mu_0(2\pi Z - h) + \mu h} \\ &\approx \frac{\mu\mu_0 NI}{\mu_0 2\pi Z + \mu h} \ . \end{aligned}$$

The current I in the wire is $I = \frac{V}{R}$, giving the final answer

$$B = \frac{\mu\mu_0 NV}{R(2\mu_0\pi Z + \mu h)} \ .$$