QUALIFYING EXAMINATION, Part 1

Solutions

Problem 1: Mathematical Methods

(a)

$$|I\rangle = \frac{|1\rangle}{\sqrt{\langle 1|1\rangle}} \equiv \frac{|1\rangle}{|1|} \ .$$

$$|II\rangle = \frac{|2\rangle - |I\rangle\langle I|2\rangle}{|numerator|} \ .$$

$$|III\rangle = \frac{|3\rangle - |I\rangle\langle I|3\rangle - |II\rangle\langle II|3\rangle|}{|numerator|} \ .$$

(b)

$$f_0 = \frac{1}{\sqrt{2}} \ .$$

No need to project out f_0 from f_1 since parity is opposite and their scalar product is zero. Thus

$$f_1 = \frac{x}{\sqrt{\int x^2 dx}} = \sqrt{\frac{3}{2}} \ .$$

To find f_2, we project out f_0 and normalize to get

$$f_2 = \sqrt{\frac{5}{2}} \cdot \frac{1}{2}(3x^2 - 1) \ .$$

(c) Without Lagrange multipliers: use $V = \pi R^2 H$ to express E as a function of H (at constant V)

$$E = \frac{C^2}{V/(\pi H)} + \frac{\pi^2}{H^2}$$

Then minimize wrt H to find

$$\frac{H}{R} = \frac{\sqrt{2}\pi}{C} \ .$$

With Lagrange multiplier: introduce a Lagrange multiplier λ and minimize

$$f = \left(\frac{C^2}{R^2} + \frac{\pi^2}{H^2}\right) - \lambda \pi R^2 H$$

wrt H and R:

$$0 = \frac{\partial f}{\partial R} = -\frac{2C^2}{R^3} - 2\lambda\pi RH \ ,$$

$$0 = \frac{\partial f}{\partial H} = -\frac{2\pi^2}{H^3} - 2\lambda\pi R^2 \ .$$

Eliminate λ and get

$$\frac{H}{R} = \frac{\sqrt{2}\pi}{C} \ .$$

(d) Expand as follows

$$|\psi\rangle = \sum_{n=1}^{N} c_n |\psi_n\rangle \quad \text{where } \textstyle\sum_{n=1}^{N} |c_n|^2 = 1 \ .$$

We then have

$$\langle\psi|A|\psi\rangle = \sum_{n=1}^{N}\sum_{m=1}^{N} c_n^* c_m \langle\psi_n|A|\psi_m\rangle = \sum_{n=1}^{N}\sum_{m=1}^{N} c_n^* c_m \lambda_m \delta_{mn} = \sum_{n=1}^{N} |c_n|^2 \lambda_n \ .$$

The sum is less than or equal to (greater than or equal to) than what we get if we replace every λ_n by λ_N (λ_1).

Problem 2: Classical Mechanics

(a)

$$M = \int \rho(r) dV = s \int_0^{2\pi} d\theta \int_0^R (kr^2) r dr = \frac{1}{2}\pi s k R^4 \ .$$

$$I = \int \rho(r) r^2 dV = s \int_0^{2\pi} d\theta \int_0^R (kr^4) r dr = \frac{1}{3}\pi s k R^6 = \frac{2}{3} M R^2 \ .$$

(b) Writing the angular velocity as $\omega = v/R$, we have

$$T = \frac{1}{2} I \omega^2 = \frac{1}{2}\left(\frac{2}{3} M R^2\right)\left(\frac{v}{R}\right)^2 = \frac{1}{3} M v^2 \ .$$

(c) The kinetic energy of the system is

$$T = \frac{1}{2}\left(m_1 + m_2 + \frac{2}{3}M\right)\dot{x}^2 \ .$$

The potential energy of the system is

$$V = -m_1 g x + m_2 g x \ .$$

Thus, the Lagrangian is given by

$$L = T - V = \frac{1}{2}\left(m_1 + m_2 + \frac{2}{3}M\right)\dot{x}^2 + (m_1 - m_2) g x \ .$$

The Lagrangian is independent of t, and therefore

$$dH/dt = \partial H/\partial t = -\partial L/\partial t = 0 \ ,$$

where H is the Hamiltonian. Thus H is a constant of the motion. Since the potential is velocity-independent, the total energy $E = H$ and the energy is conserved. Can also use Noether's theorem.

(d) The Euler-Lagrange equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \ .$$

Inserting the Lagrangian, this gives

$$\left(m_1 + m_2 + \frac{2}{3}M\right)\ddot{x} = (m_1 - m_2) g \ ,$$

so the acceleration is constant and given by

$$a = \ddot{x} = \frac{(m_1 - m_2) g}{m_1 + m_2 + \frac{2}{3}M} \ .$$

(e) The tensions can be found by balancing the forces on each of the blocks

$$m_1 a = m_1 g - T_1$$
$$m_2(-a) = m_2 g - T_2$$

This gives

$$T_1 = m_1(g-a) = \frac{m_1 g(2m_2 + \frac{2}{3}M)}{m_1 + m_2 + \frac{2}{3}M}$$
$$T_2 = m_2(g+a) = \frac{m_2 g(2m_1 + \frac{2}{3}M)}{m_1 + m_2 + \frac{2}{3}M}$$

The net torque on the pulley is

$$\tau = (T_1 - T_2)R = I\left(\frac{a}{R}\right) = \frac{2}{3}MR\frac{(m_1 - m_2)g}{m_1 + m_2 + \frac{2}{3}M} \ .$$

Problem 3: Electromagnetism I

(a) Since the cylinder is infinitely long and the external field is uniform, the system is invariant under a displacement in the z direction. Thus the electrostatic potential must be independent of z, i.e., $\frac{\partial V}{\partial z} = 0$, and the 3-D Laplace equation in cylindrical coordinates reduces to the 2-D Laplace's equation in polar coordinates r, ϕ.

In the following we choose the x axis along the direction of the electric field. The boundary conditions on $V(r, \phi)$ are then:

- The solution is periodic in ϕ with period of 2π (the potential is single-valued in ϕ).
- The electric field approaches $\vec{E}_0$ far away from the cylinder ($r \to \infty$). For $\vec{E}_0 = E_0 \hat{x}$, the potential is given by $-E_0 x = -E_0 r \cos\phi$. Thus the boundary condition at $r \to \infty$ is
$$V(r, \phi) \to -E_0 r \cos\phi \ .$$
- The potential is zero at $r = R$: $V(R, \phi) = 0$ (the cylinder is grounded).
- The potential is finite for $r \to 0$.

(b) The general solution to Laplace's equation in polar coordinates is given by

$$V(r, \phi) = a_0 + b_0 \ln r + \sum_{m=1}^{\infty} (a_m r^m + b_m r^{-m})[A_m \sin(m\phi) + B_m \cos(m\phi)] \ ,$$

where a_m, b_m, A_m, B_m are constants. The boundary condition in ϕ are already satisfied by the choice of $\pm m$ being integers. To determine V_{out}, we use the boundary conditions in (a):

- The condition $V_{\text{out}}(r, \phi) \to -E_0 r \cos\phi$ for $r \to \infty$ leads to:
  $a_0 = b_0 = 0$ and only the $m = 1$ term in the sum over m survives.
  $A_1 = 0,\ B_1 = 1;\ \ a_1 = -E_0$.
- The potential is zero at $r = R$:
$$V_{\text{out}}(R, \phi) = (-E_0 R + b_1/R)\cos\phi = 0 \Longrightarrow b_1 = E_0 R^2 \ .$$

Thus, the external electric field at $r > R$ is given by

$$V_{\text{out}}(r, \phi) = -E_0 r \cos\phi \left(1 - \frac{R^2}{r^2}\right) \ .$$

(c) Inside the cylinder ($r < R$), the electric field is zero, and the potential is constant. Since the conducting cylinder is grounded, the potential is zero everywhere inside the cylinder.

One can also consider Laplace's equation for the potential $V_{\text{in}}(r, \phi)$ inside the cylinder with the boundary conditions that $V_{\text{in}}(R, \phi) = 0$ and V_{in} must be finite for $r \to 0$. Since the solution $V = 0$ satisfies these boundary conditions and the solution to Laplace's equation with given boundary conditions is unique, it follows that $V_{\text{in}} = 0$.

(d) According to the Gauss' law, the induced charge density is given in terms of the normal component of the electric-field; i.e., $E_n = 4\pi\sigma$ at $r = R$. Therefore,

$$\sigma = \frac{1}{4\pi}\left(-\frac{\partial V_{\text{out}}}{\partial r}\right)|_{r=R} = \frac{1}{4\pi}E_0\cos\phi\left(1 + \frac{R^2}{r^2}\right)|_{r=R} = \frac{E_0\cos\phi}{2\pi}\ .$$

Problem 4: Electromagnetism II

(a) This is just a simple harmonic oscillator with a driving force (given by the Lorentz force):

$$m\ddot{\mathbf{r}} = -m\omega_0^2\mathbf{r} - e\mathbf{E} - \frac{eB_0}{c}(\dot{\mathbf{r}} \times \hat{\mathbf{z}}) \ .$$

(b) Substituting

$$\mathbf{r} = \mathbf{r}_0 e^{i(kz-\omega t)}$$

in the equation of motion in part (a), we find

$$-m\omega^2\mathbf{r}_0 = -m\omega_0^2\mathbf{r}_0 - e\mathbf{E}_0 + i\frac{e\omega B_0}{c}(\mathbf{r}_0 \times \hat{\mathbf{z}}) \ .$$

Decomposing $\mathbf{r}_0$ and $\mathbf{E}_0$ into their $\pm$ components, we have

$$\mathbf{r}_0 = r_0^-\mathbf{e}^+ + r_0^+\mathbf{e}^- \ ,$$

$$\mathbf{E}_0 = E_0^-\mathbf{e}^+ + E_0^+\mathbf{e}^- \ .$$

This gives us two decoupled equations

$$-\omega^2 r_0^\pm = -\omega_0^2 r_0^\pm - \frac{e}{m}E_0^\pm \pm \frac{e\omega B_0}{mc}r_0^\pm \ .$$

Substituting the definition of the cyclotron frequency $\omega_c = \frac{eB_0}{mc}$ and combining like terms, we find

$$r_0^\pm(\omega_0^2 - \omega^2 \mp \omega_c\omega) = -\frac{eE_0^\pm}{m} \ .$$

Thus

$$r_0^\pm = -\frac{eE_0^\pm}{m}\frac{1}{\omega_0^2 - \omega^2 \mp \omega_c\omega} \ .$$

(c) The polarization is

$$\mathbf{P} = -Ne\mathbf{r} \ .$$

Recalling that (in Gaussian units) $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \epsilon\mathbf{E}$, we get the two dielectric constants

$$\epsilon^\pm = 1 + \frac{4\pi Ne^2}{m}\frac{1}{\omega_0^2 - \omega^2 \mp \omega_c\omega} \ .$$

(d) The velocity of light in a dielectric medium is proportional to $1/\sqrt{\epsilon}$. The indices of refraction are $n^{\pm} = \sqrt{\epsilon^{\pm}}$, where we ignore the correction to the magnetic permeability. Substituting the result from part (c), we find

$$n^{\pm} = \sqrt{1 + \frac{4\pi N e^2}{m} \frac{1}{\omega_0^2 - \omega^2 \mp \omega_c \omega}}\,. \tag{1}$$