QUALIFYING EXAMINATION, Part 2

Solutions

Problem 1: Quantum Mechanics I

(a) Degeneracy $= 2(2l+1)$

This degeneracy is due to the rotational symmetry of the system, with respect to orbital and spin angular momentum independently.

$j = l \pm \frac{1}{2}$, except for $l = 0$, in which case $j = 1/2$.

(b) Using $\vec{L} \cdot \vec{S} = \frac{1}{2}(\vec{J}^2 - \vec{L}^2 - \vec{S}^2)$, we have

$$\langle \vec{L} \cdot \vec{S} \rangle = \frac{\hbar^2}{2}[j(j+1) - l(l+1) - 3/4]$$

Thus

$\langle \vec{L} \cdot \vec{S} \rangle = (\hbar^2/2)l \quad \text{for} \quad j = l + 1/2$

$\langle \vec{L} \cdot \vec{S} \rangle = -(\hbar^2/2)(l+1) \quad \text{for} \quad j = l - 1/2$

(c) Using the results in (b)

$$\langle H_{LS} \rangle = \frac{\hbar^2}{4mc^2} l \left\langle \frac{1}{r}\frac{dV}{dr} \right\rangle_{nl} \quad \text{for} \quad j = l + \frac{1}{2},$$

$$\langle H_{LS} \rangle = -\frac{\hbar^2}{4mc^2}(l+1) \left\langle \frac{1}{r}\frac{dV}{dr} \right\rangle_{nl} \quad \text{for} \quad j = l - \frac{1}{2}.$$

The degeneracy of each of the two levels is $2j+1$. This degeneracy is $2l+2$ for $j = l + 1/2$, and $2l$ for $j = l - 1/2$.

The degeneracy remains due to the rotational symmetry of the system with respect to total angular momentum.

(d) The 3s and 3p states are not degenerate since the potential seen by the outer electron is not pure Coulombic. The 3p level is split by the fine structure into two levels: four $j = 3/2$ states and two $j = 1/2$ states. The $j = 3/2$ level lies above the $j = 1/2$ level.

Problem 2: Quantum Mechanics II

(a) We have

Defining $D(-\alpha) = e^A$ with $A = -\alpha a^\dagger + \alpha^* a$, we have

$$D(-\alpha) = e^{-A^\dagger} = (e^{-A})^\dagger = D^\dagger(\alpha) \ .$$

(This is equivalent to the unitarity of D). Using

$$e^A X e^{-A} = X + [A, X] + \frac{1}{2!}[A, [A, X]] + \dots$$

with $A = -\alpha a^\dagger + \alpha^* a$ and $X = a$, and

$$[-\alpha a^\dagger + \alpha^* a, a] = \alpha \ ,$$

we find

$$\hat{D}^\dagger(\alpha)\, \hat{a}\, \hat{D}(\alpha) = \hat{a} + \alpha \ .$$

(b) Using the hint, we find

$$e^{\alpha a^\dagger} e^{-\alpha^* a} = e^{\alpha a^\dagger - \alpha^* a + \frac{1}{2}|\alpha|^2} = e^{\frac{1}{2}|\alpha|^2} D(\alpha) \ .$$

Thus

$$|\alpha\rangle = D(\alpha)|0\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} e^{-\alpha^* a}|0\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger}|0\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_n \frac{1}{n!}\alpha^n (a^\dagger)^n |0\rangle = \sum_n c_n |n\rangle \ ,$$

where

$$c_n = \frac{\alpha^n e^{-\frac{1}{2}|\alpha|^2}}{\sqrt{n!}}$$

and we have used $|n\rangle = \frac{a^{\dagger n}}{\sqrt{n!}}$. Note that $|c_n|^2 = \frac{\lambda^n e^{-\lambda}}{n!}$ is a Poisson distribution with $\lambda = |\alpha|^2$.

(c) First method [using (b)]:

$$a|\alpha\rangle = a\left(e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0} \frac{1}{\sqrt{n!}}\alpha^n |n\rangle\right) = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0} \frac{1}{\sqrt{n!}}\alpha^n a|n\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=1} \frac{1}{\sqrt{(n-1)!}}\alpha^n |n-1\rangle \ ,$$

where we have used $a|n\rangle = \sqrt{n}|n-1\rangle$. Relabeling $n-1$ by n in the sum in the last expression on the r.h.s., we find

$$a|\alpha\rangle = \alpha\left(e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0} \frac{1}{\sqrt{n!}}\alpha^n |n\rangle\right) = \alpha|\alpha\rangle \ .$$

Second method [using (a)]:

$$a|\alpha\rangle = aD(\alpha)|0\rangle = [D(\alpha)D^\dagger(\alpha)]aD(\alpha)|0\rangle = D(\alpha)[D^\dagger(\alpha)aD(\alpha)]|0\rangle = D(\alpha)(a+\alpha)|0\rangle = \alpha|\alpha\rangle \ ,$$

where we have used the unitarity of $D(\alpha)$ and $a|0\rangle = 0$.

(d) First method: using the result in (b), the state at time t is

$$e^{-\frac{i}{\hbar}Ht}|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}\frac{1}{\sqrt{n!}}\alpha^n e^{-\frac{i}{\hbar}Ht}|n\rangle \ .$$

Using $H|n\rangle = (n+1/2)\hbar\omega$, we find

$$e^{-\frac{i}{\hbar}Ht}|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}\frac{1}{\sqrt{n!}}\alpha^n e^{-i\omega(n+1/2)t}|n\rangle = e^{-\frac{i}{2}\omega t}e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}\frac{1}{\sqrt{n!}}\alpha^n(e^{-i\omega t})^n|n\rangle \ .$$

This can be written as

$$e^{-\frac{i}{\hbar}Ht}|\alpha\rangle = e^{-\frac{i}{2}\omega t}e^{-\frac{1}{2}|\alpha e^{-i\omega t}|^2}\sum_{n=0}\frac{1}{\sqrt{n!}}(\alpha e^{-i\omega t})^n|n\rangle = e^{-\frac{i}{2}\omega t}|\alpha(t)\rangle \ ,$$

where

$$\alpha(t) = e^{-i\omega t}\alpha \ .$$

Second method: denoting the time evolution by $U(t) = e^{-\frac{i}{\hbar}Ht}$, the state at time t is

$$U|\alpha\rangle = UD(\alpha)|0\rangle = [UD(\alpha)U^\dagger]U|0\rangle \ .$$

To calculate $UD(\alpha)U^\dagger$, we use $[H,a] = -\hbar\omega a$ to find $UaU^\dagger = e^{i\omega t}a$ and $Ua^\dagger U^\dagger = e^{-i\omega t}a^\dagger$. We then have

$$UD(\alpha)U^\dagger = Ue^{\alpha a^\dagger - \alpha^* a}U^\dagger = e^{\alpha e^{-i\omega t}a^\dagger - \alpha^* e^{i\omega t}a} = D(\alpha(t)) \ ,$$

where $\alpha(t) = e^{-i\omega t}\alpha$. Thus

$$U|\alpha\rangle = D(\alpha(t))U|0\rangle = e^{-\frac{i}{2}\omega t}|e^{-i\omega t}\alpha\rangle \ .$$

Statistical Mechanics I

(a) The Hamiltonian of the particle is

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + \frac{1}{2} m\omega^2 z^2 \ .$$

The classical partition function is

$$Z = \int dx\, dy\, dz\, dp_x\, dp_y\, dp_z\, e^{-\beta H} \ .$$

where $\beta = 1/(kT)$ (k is the Boltzmann constant). The x, y integrals give the area A, while each of the z and momenta integrals is a Gaussian. We find

$$Z = A\,(2\pi m/\beta)^{3/2} \left[2\pi/\beta(m\omega^2)\right]^{1/2} = A\,(2\pi/\beta)^2\,(m/\omega) = A\,(2\pi kT)^2\,(m/\omega) \ .$$

The energy is calculated from

$$E = -\partial \ln Z/\partial\beta = \frac{2}{\beta} = 2kT \ .$$

The free energy is

$$F(T) = -kT \ln Z = -kT\left[\ln A + 2\ln(2\pi kT) + \ln(m/\omega)\right] \ .$$

The entropy is calculated from $S = (E - F)/T$ to be

$$S(T) = k\left[2 + \ln A + 2\ln(2\pi kT) + \ln(m/\omega)\right] \ .$$

The entropy can also be calculated from $S = -(\partial F/\partial T)_A$.

We have four quadratic degrees of freedom in the Hamiltonian, each contributing $kT/2$. Using the equipartition theorem $E = 2kT$.

(b) We have a simple harmonic oscillator along the z direction with energy level spacing of $\hbar\omega$. Quantum effects become important when $kT < \hbar\omega$. We estimate T_1 from $kT_1 = \hbar\omega$ or

$$T_1 = \frac{\hbar\omega}{k} \ .$$

Regarding the motion in the x-y plane, quantum effects (having to do with the finite size of A) become important when the thermal energy kT is lower than the energy spacing for the particle-in-a-box. The energy scale for particle-in-a-box is set by $p_x = p_y = \hbar\pi/L$ where $L = \sqrt{A}$ is the length of the sides of the square. The associated energy is $\hbar^2\pi^2/(mL^2)$, and T_2 is estimated by

$$T_2 = \frac{\hbar^2\pi^2}{mAk} \ .$$

(c) For temperatures $T_2 \ll T < T_1$, we can still treat the x-y motion classically but the z direction is quantum mechanical with energy levels of $(n + 1/2)\hbar\omega$ $(n = 0, 1, 2, \ldots)$. The partition function is then

$$Z = \int dx \int dy \int dp_x \int dp_y e^{-\beta(p_x^2+p_y^2)/(2m)} \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+1/2)} ,$$

or

$$Z = A(2\pi m/\beta) \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega}} .$$

The average energy is then given by

$$E = -\frac{\partial \ln Z}{\partial \beta} = kT + \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\hbar\omega/(kT)} - 1} .$$

The first term is the equipartition contribution from the kinetic energies in x-y, the second term is the zero-point energy of the quantum oscillator, and the third is the thermal excitation energy of the oscillator.

(d) For $T = 0$ the particle is in its quantum ground state in both the z and x-y motion.

$$E = \frac{\hbar\omega}{2} + \frac{\hbar^2\pi^2}{mA} .$$

The ground-state is non-degenerate so the entropy $S = 0$.

Problem 4: Statistical Mechanics II

(a) Assuming the neutrinos are massless, their dispersion relation is given by $\epsilon = \hbar ck$. Since the neutrinos have 2 spin states, their density of single-particle states in momentum space is given by $g(\mathbf{k}) = 2V/(2\pi)^3$. We have $2\frac{V}{(2\pi)^3}d^3\mathbf{k} = 2\frac{V}{(2\pi)^3}4\pi k^2\,dk = \frac{V}{\pi^2}k^2\frac{dk}{d\epsilon}d\epsilon = \frac{V}{\pi^2}\frac{\epsilon^2}{(\hbar c)^3}d\epsilon$, where we have used $d\epsilon = \hbar c\,dk$. Therefore

$$g(\epsilon) = \frac{V}{\pi^2(\hbar c)^3}\epsilon^2 \ .$$

The photons are massless particles with two polarization states and thus have the same density of states as the neutrinos.

(b) The average number of neutrinos at temperature T is determined by the equilibrium condition $(\partial F/\partial N)_{T,V} = 0$ where F is the free energy. Since the chemical potential $\mu = (\partial F/\partial N)_{T,V}$ it follows that $\mu = 0$.

Since neutrinos are fermions, the average occupation number of a single-particle state with energy ϵ is $n(\epsilon) = (1 + e^{\beta\epsilon})^{-1}$. The total energy is then given by

$$E = \int_0^\infty d\epsilon g(\epsilon)\epsilon n(\epsilon) = \frac{V}{\pi^2(\hbar c)^3}\int_0^\infty d\epsilon\frac{\epsilon^3}{1+e^{\beta\epsilon}} \ .$$

Substituting $x = \beta\epsilon$, we find

$$E = \frac{V}{\pi^2(\hbar c)^3}\frac{1}{\beta^4}\left(\int_0^\infty dx\frac{x^3}{e^x+1}\right) \ .$$

The energy density of the neutrino gas is thus given by

$$\frac{E}{V} = \frac{k^4}{\pi^2(\hbar c)^3}\left(\int_0^\infty dx\frac{x^3}{e^x+1}\right)T^4 \propto T^4 \ ,$$

where the dimensionless definite integral in brackets is a temperature-independent constant

$$\int_0^\infty dx\frac{x^3}{e^x+1} = \frac{7}{8}\Gamma(4)\zeta(4) \ .$$

(c) The energy of the photon gas is given by a similar expression to that of the neutrino gas except that the photons are bosons and $n(\epsilon) = (e^{\beta\epsilon} - 1)^{-1}$. We have

$$\frac{E}{V} = \frac{k^4}{\pi^2(\hbar c)^3}\left(\int_0^\infty dx\frac{x^3}{e^x-1}\right)T^4 \ ,$$

where

$$\int_0^\infty dx \frac{x^3}{e^x - 1} = \Gamma(4)\zeta(4) \; .$$

The ratio between the two energy densities is then given by the ratio between the two definite integrals, i.e., 7/8.

(d) The pressure of non-interacting identical particles is given by

$$p = \sum_r \left(-\frac{\partial \epsilon_r}{\partial V} \right) \langle n_r \rangle \; ,$$

where ϵ_r is the energy of single-particle state r and $\langle n_r \rangle$ is the occupation of state r.

The dispersion relation for the neutrino is $\epsilon = c\hbar k$. Using the quantization condition of k in a box of side L, $k_i = (2\pi/L)n_i$ $(i = 1, 2, 3)$ (periodic b.c.), we find $\epsilon \propto V^{-1/3}$. It follows

$$-\frac{\partial \epsilon}{\partial V} = \frac{1}{3}\frac{\epsilon}{V} \; .$$

Using this relation in the above expression for p, we find

$$p = \frac{1}{3V} \sum_r \epsilon_r \langle n_r \rangle = \frac{1}{3}\frac{E}{V} \; .$$