QUALIFYING EXAMINATION, Part 1

Solutions

Problem 1: Mathematical Methods

(a) For a series $S = \sum_n a_n$ to converge, it is necessary that the absolute value of the ratio a_{n+1}/a_n for large n be < 1. We have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(5x)^{n+1}}{(5x)^n} \frac{(n+1)^2+1}{n^2+1} \right| = |5x| < 1$$

So we need $|x| < 1/5$. Thus the radius of convergence is $1/5$.

The series also converges at $x = 1/5$ where it equals $\sum 1/(n^2+1)$ and is integrable.

(b) Since the integrand is even, we can extend the lower limit to $-\infty$ and divide by 2. Since $\cos x = Re\ [e^{ix}]$, and relabeling x as z, the integral is given by

$$I = \frac{1}{2} Re \int_{-\infty}^{\infty} \frac{e^{iz}}{1+z^2} dz \ .$$

We close the contour counterclockwise by a semicircle in the upper half plane with large radius $R \to \infty$. The integral over the semicircle vanishes since for $z = x + iy$ in the upper half plane $(y > 0)$

$$\left| \frac{e^{iz}}{1+z^2} \right| = \frac{e^{-y}}{|1+z^2|} \leq \frac{1}{|1+z^2|} \sim \frac{1}{R^2} \ ,$$

and $\pi R \times \frac{1}{R^2} \to 0$. Denoting the close contour by $\mathcal{C}$, we have

$$I = \frac{1}{2} Re \int_{\mathcal{C}} \frac{e^{iz} dz}{1+z^2} dz = \frac{1}{2} Re \int_{\mathcal{C}} \frac{e^{iz} dz}{(z+i)(z-i)} dz \ .$$

The integrand has two simple poles at $z = \pm i$, and only $z = i$ is enclosed by the contour. Using the residue theorem, we find

$$I = \frac{1}{2} Re \left[2\pi i \left. \frac{e^{iz}}{z+i} \right|_{z=i} \right] = \frac{\pi}{2e} \ .$$

(c) Assume that $\vec{V}$ and $\vec{V}'$ are two vector fields with the same curl and divergence. Then their difference $\vec{W} = \vec{V} - \vec{V}'$ has zero curl and zero divergence

$$\nabla \times \vec{W} = 0 \ ; \quad \nabla \cdot \vec{W} = 0 \ ,$$

and $\vec{W} = 0$ at infinity. Since $\nabla \times \vec{W} = 0$, there is a scalar field ϕ such that

$$\vec{W} = \nabla\phi \ .$$

Since $\nabla \cdot \vec{W} = 0$, and $\nabla \cdot \vec{W} = \nabla \cdot \nabla\phi = \nabla^2\phi$, we have

$$\nabla^2\phi = 0 \ .$$

Next we integrate the identity in the hint over all of space

$$\int \nabla \cdot (\phi\nabla\phi) \ dV = \int \ |\nabla\phi|^2 \ dV + \int \phi\nabla^2\phi \ dV = \int \ |\nabla\phi|^2 \ dV \ ,$$

where we have used $\nabla^2\phi = 0$. Using the divergence theorem, we can convert the integral on the l.h.s. to a surface integral

$$\int_{\text{surface at infinity}} \phi\nabla\phi \cdot dS = \int |\nabla\phi|^2 \ dV \ .$$

Since $\vec{W} = \nabla\phi = 0$ at infinity, the l.h.s. vanishes. Since $|\nabla\phi|^2 \geq 0$, it follows that $\nabla\phi = \vec{W} = 0$ and thus $\vec{V} = \vec{V}'$.

Second method: the scalar field satisfies Laplace equation $\nabla^2\phi = 0$ and the boundary condition $\phi =$ const at infinity (since $\vec{W} = \nabla\phi = 0$ at infinity). Since the solution to Laplace equation with given boundary conditions is unique, we have $\phi =$ const in all of space. Thus $\vec{W} = \nabla\phi = 0$.

Problem 2: Classical Mechanics

(a) In the limit of small φ, the coordinates x, y of the Huygens pendulum are given by

$$x = \frac{l}{4}(\varphi + \sin\varphi) \approx \frac{l}{4}(\varphi + \varphi) = l\varphi/2,$$

$$y = \frac{l}{4}(3 + \cos\varphi) \approx \frac{l}{4}\left(3 + 1 - \varphi^2/2\right) = l - l\varphi^2/8.$$

These values of x, y are the same as for the simple pendulum in the limit of small θ

$$x = l\sin\theta \approx l\varphi/2\ .$$

$$y = l\cos\theta \approx l(1 - \varphi^2/8)\ .$$

(b) The velocities $\dot{x}, \dot{y}$ are given by

$$\begin{aligned} \dot{x} &= \frac{l}{4}(\dot{\varphi} + \dot{\varphi}\cos\varphi) \\ \dot{y} &= -\frac{l}{4}\dot{\varphi}\sin\varphi \end{aligned}$$

so the kinetic energy is

$$\begin{aligned} T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) &= \tfrac{1}{2}m\left(\frac{l}{4}\right)^2 \dot{\varphi}^2\left(1 + 2\cos\varphi + \cos^2\varphi + \sin^2\varphi\right) \\ &= m\left(\frac{l}{4}\right)^2 \dot{\varphi}^2(1 + \cos\varphi) \end{aligned}$$

The potential energy is

$$V = -mg\frac{l}{4}(3 + \cos\varphi)\ .$$

Thus the Lagrangian is

$$\mathcal{L} = T - V = m\frac{l}{4}\left[\frac{l}{4}\dot{\varphi}^2(1 + \cos\varphi) + g(3 + \cos\varphi)\right]\ .$$

(c) Using $u = \sin(\varphi/2)$, we find

$$\dot{\varphi} = \frac{2\dot{u}}{\sqrt{1 - u^2}}\ ; \quad \cos\varphi = 1 - 2u^2\ .$$

Substituting in the Lagrangian in (b), we find

$$\mathcal{L} = m\frac{l}{2}\left(l\dot{u}^2 + 2g - gu^2\right)\ .$$

(d) We have

$$\frac{\partial \mathcal{L}}{\partial \dot{u}} = ml^2\dot{u}\ , \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{u}} = ml^2\ddot{u}$$

$$\frac{\partial \mathcal{L}}{\partial u} = -mglu.$$

Thus, the Euler-Lagrange equation for u is

$$\ddot{u} + \frac{g}{l}u = 0\ .$$

This is the equation of an harmonic oscillator and its general solution is given by

$$u = u_0 \sin(\omega t + \theta_0)\ .$$

where $\omega = \sqrt{\frac{g}{l}}$. The angle $\varphi = 2 \arcsin u$ is thus periodic in time with a period of $T = 2\pi/\omega = 2\pi\sqrt{l/g}$, the same as for the simple pendulum in the limit of small oscillations.

Problem 3: Electromagnetism I

(a) The system is invariant under rotations around the direction of the external electric field (i.e., around the z axis). The electrostatic potential is then independent of the azimuthal angle ϕ, i.e., $V = V(r, \theta)$.

There are no free or bound charges inside the volume of the sphere (there are only bound charges on the surface), so $V_{\rm in}$ satisfies Laplace's equation. The general solution of Laplace's equation is given in the hint. Since $V_{\rm in}(r, \theta)$ is finite at $r = 0$, we have

$$V_{\rm in}(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \ .$$

(b) The electric field approaches $\vec{E}_0$ far away from the sphere ($r \gg R$). For $\vec{E}_0 = E_0\hat{z}$, the potential is given by $-E_0 z = -E_0 r\cos\theta$. Thus the boundary condition is

$$V_{\rm out} \to -E_0 r\cos\theta \quad (\text{for} \ \ r \gg R) \ .$$

Using the general solution in the hint together with the above boundary condition, we have

$$V_{\rm out}(r, \theta) = -E_0 r\cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \ .$$

(c) At $r = R$, the potential is continuous

$$V_{\rm in}(R, \theta) = V_{\rm out}(R, \theta) \ ,$$

and the normal component of the displacement field $\vec{D} = \varepsilon\vec{E}$ is continuous (since there is no free charge at the surface)

$$\varepsilon\frac{\partial V_{\rm in}}{\partial r} = \varepsilon_0 \frac{\partial V_{\rm out}}{\partial r} \quad (\text{at} \ \ r = R) \ .$$

Using the general form of $V_{\rm in}$ and $V_{\rm out}$ in (b) and (c), and equating the coefficients of P_l, the first boundary condition gives

$$\begin{aligned} A_1 R &= -E_0 R + \frac{B_1}{R^2} \quad (l = 1) \\ A_l R^l &= \frac{B_l}{R^{l+1}} \quad (l \neq 1) \ , \end{aligned}$$

while the second boundary condition (using $\varepsilon_r = \varepsilon/\epsilon_0$) gives

$$\begin{aligned} \varepsilon_r A_1 &= -E_0 - \frac{2B_1}{R^3} \quad (l = 1) \\ \varepsilon_r A_l l R^{l-1} &= -\frac{(l+1)B_l}{R^{l+2}} \quad (l \neq 1) \ . \end{aligned}$$

Using the $l \neq 1$ equations, we find

$$A_l = B_l = 0 \quad (l \neq 1) \; ,$$

while the $l = 1$ equations give

$$\begin{aligned} A_1 &= -\frac{3}{\varepsilon_r + 2} E_0 \\ B_1 &= \frac{\varepsilon_r - 1}{\varepsilon_r + 2} R^3 E_0 \; . \end{aligned}$$

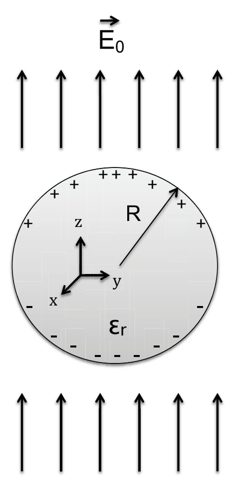
The potential inside the sphere is then

$$V_{\text{in}}(r, \theta) = -\frac{3}{\varepsilon_r + 2} E_0 r \cos\theta = -\frac{3}{\varepsilon_r + 2} E_0 z \; .$$

The electric field inside the sphere is thus uniform

$$\vec{E}_{\text{in}} = -\nabla V_{\text{in}} = \frac{3}{\varepsilon_r + 2} \vec{E}_0 \; .$$

(d)



The induced bound charge generates an electric field inside the dielectric sphere that is in an opposite direction to $\vec{E}_0$, thus reducing the total field inside the sphere to $\frac{3}{\varepsilon_r + 2}\vec{E}_0$ ($\epsilon_r > 1$).

Problem 4: Electromagnetism II

(a) The magnetic field inside the solenoid must, by symmetry, be parallel to the axis of the solenoid. Consider a rectangular Amperian loop of length ℓ with one side outside the solenoid, we find

$$B\ell = \mu_0 n I \ell$$
$$B = \mu_0 n I \, .$$

(b) By symmetry, the electric field must be in the radial direction. We apply Gauss law to a cylinder of radius r and length $\ell < l$. If $r > a$, the net charge inside is zero, and $E = 0$. For $r < a$

$$E 2\pi r \ell = \epsilon_0 \frac{Q}{l} \ell$$
$$E = \frac{Q}{2\pi \epsilon_0 l r} \, .$$

(c) The induced electric field E_{ind} is along the circumference. Apply Faraday's law to a loop along the circumference of the cylinder of radius a

$$E_{\text{ind}} 2\pi a = \frac{d\Phi_b}{dt} \, ,$$

where $\Phi_b = B\pi a^2$ is the magnetic flux. We find

$$E_{\text{ind}} = \frac{1}{2} \mu_0 n a \frac{dI}{dt} \, .$$

(d) Consider a patch of area dA on the surface of the cylinder with charge dQ; The tangential force due to the induced electric field on the patch is $dF = E_{\text{ind}} \, dQ$. The torque $d\vec{\tau} = \vec{r} \times d\vec{F}$ is along the axis of the cylinder and its magnitude is $d\tau = \frac{1}{2} \mu_0 n a^2 \frac{dI}{dt} \, dQ$. Note that this is uniform across the entire surface of the cylinder. The total torque on the cylinder is

$$\tau = \frac{1}{2} \mu_0 n a^2 Q \frac{dI}{dt} \, ,$$

causing the cylinder to rotate. Denoting by $\vec{L}$ the angular momentum of the cylinder, we have $d\vec{L}/dt = \vec{\tau}$. The angular momentum of the cylinder at any time after the current in the solenoid begins to decrease is then given by $\vec{L} = \int \vec{\tau} dt$. $\vec{L}$ is along the symmetry axis of the cylinder, and its magnitude L once the current becomes zero is

$$L = \frac{1}{2}\mu_0 n a^2 Q \int \frac{dI}{dt} dt = \frac{1}{2}\mu_0 n a^2 Q I \ .$$

When the current in the solenoid decreases, the flux through the cylinder decreases, and according to Lenz's law, the direction of the induced emf in the cylinder is to oppose this. Thus the direction of the induced electric field is the same as the direction of the current (so as to drive a current that will increase the flux in the cylinder). Since the charge on the cylinder is negative, the force on the cylinder is opposite to the direction of the induced electric field. Thus the cylinder rotates in a direction opposite to the direction of the current in the solenoid.