QUALIFYING EXAMINATION, Part 1

Solutions

Problem 1: Mathematical Methods

(a) (i) The function $\sin x$ on the l.h.s. of the formula has zeros at $x = m\pi$ $(m = 0, 1, 2, \ldots)$. $x = 0$ is a zero of the r.h.s. because of the factor x. For $x = m\pi$ $(m = 1, 2, \ldots)$ to be a zero of the r.h.s., we require that $na = m\pi$ for at least one value of n. This condition is satisfies (for each m) by choosing $a = \pi$ and $n = m$.

(ii) Set $x = \frac{\pi}{2}$ in the formula to obtain

$$1 = \frac{\pi}{2}\prod_{1}^{\infty}\left(1 - \frac{1}{4n^2}\right)$$

so the answer is $\frac{2}{\pi}$.

(iii) Take $x \to \pi$ to find

$$\lim_{x\to\pi}\frac{\sin x}{x(1-\frac{x^2}{\pi^2})} = \prod_{n=2}^{\infty}\left(1 - \frac{1}{n^2}\right) \ .$$

Since $\sin x \simeq \pi - x$ near $x = \pi$ we find

$$\lim_{x\to\pi}\frac{\pi^2(\pi - x)}{\pi(\pi - x)(x + \pi)} = \prod_{n=2}^{\infty}\left(1 - \frac{1}{n^2}\right)$$

so that the answer is $\frac{1}{2}$.

(b) Take a derivative of the Cauchy formula with respect to z to find

$$\frac{df(z)}{dz} = \frac{1}{2\pi i}\oint_C \frac{f(z')dz'}{(z'-z)^2} \ .$$

The integrand in the given contour integral has a simple pole at $z = 1$ and a double pole at $z = 0$. Both poles are inside the contour. The contribution of the simple pole to the contour integral is determined from Cauchy's formula with $f(z) = \frac{e^{az}}{z^2}$ and the contribution of the double pole is determined from the derivative formula above using $f(z) = \frac{e^{az}}{(z-1)}$. We find

$$\oint_{|z|=3.141}\frac{e^{az}dz}{z^2(z-1)} = 2\pi i\left[\left(\frac{e^{az}}{z^2}\right)\Big|_1 + \frac{d}{dz}\left(\frac{e^{az}}{z-1}\right)\Big|_0\right] = 2\pi i\left(e^a - a - 1\right) \ .$$

(c) (i)

$$\frac{\partial V}{\partial x} = 2x + 2y = 0$$
$$\frac{\partial V}{\partial y} = 2x - 4y = 0$$

imply $x = y = 0$ for the only stationary point.

The potential can be written as a quadratic form

$$V(x, y) = \frac{1}{2}(x, y)\begin{pmatrix} 2 & 2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

The eigenvalues of the 2×2 potential matrix are

$$\lambda_\pm = -1 \pm \sqrt{13}\,.$$

One eigenvalue is positive and the second is negative, so it is a saddle point.

(ii) The stable direction corresponds to the positive eigenvalue

$$\lambda_+ = \sqrt{13} - 1\,.$$

The corresponding frequency (taking $m = 1$) is $\omega = \sqrt{\lambda_+} = \sqrt{\sqrt{13} - 1}$. The direction is that of the corresponding eigenvector whose components x, y satisfy

$$(2 - \lambda_+)x + 2y = (3 - \sqrt{13})x + 2y = 0\,.$$

Problem 2: Classical Mechanics

(a)

$$\sin\theta_n = \frac{x_n - x_{n+1}}{l} \ .$$

(b) Applying Newton's second law in the horizontal direction for the n-th mass, we find

$$m\ddot{x}_n = T_{n-1}\sin\theta_{n-1} - T_n \sin\theta_n \ .$$

Using the result in (a) and the hint $T_n = nmg$, we find

$$m\ddot{x}_n = nmg\left(\frac{x_{n+1} - x_n}{l}\right) - (n-1)mg\left(\frac{x_n - x_{n-1}}{l}\right) \ .$$

(c) Assuming a harmonic solution, we have $\ddot{x}_n = -\omega^2 x_n$, and the equation in (b) can be written in the form

$$-m\omega^2 x_n = nmg\left(\frac{x_{n+1} - x_n}{l}\right) - (n-1)mg\left(\frac{x_n - x_{n-1}}{l}\right) \ .$$

We then find the recurrence relation

$$x_{n+1} = \left(2 - \frac{1}{n} - \frac{l\omega^2}{ng}\right)x_n - \left(1 - \frac{1}{n}\right)x_{n-1} \ .$$

(d) We use the recurrence relation in (c) for $n = 2$ and $n = 1$. For $n = 2$, using $x_3 = 0$, we find

$$\begin{aligned} 0 &= \left(2 - \frac{1}{2} - \frac{l\omega^2}{2g}\right)x_2 - \left(1 - \frac{1}{2}\right)x_1 \\ 0 &= \left(\frac{3}{2} - \frac{l\omega^2}{2g}\right)x_2 - \frac{1}{2}x_1 \end{aligned} \ .$$

For $n = 1$, we have

$$\begin{aligned} x_2 &= \left(2 - 1 - \frac{l\omega^2}{g}\right)x_1 \\ x_2 &= \left(1 - \frac{l\omega^2}{g}\right)x_1 \end{aligned} \ .$$

Substituting the second equation into the first gives

$$0 = \left(\frac{3}{2} - \frac{l\omega^2}{2g}\right)\left(1 - \frac{l\omega^2}{g}\right)x_1 - \frac{1}{2}x_1 \ .$$

For a non-trivial solution $x_1 \neq 0$. Dividing by x_1 and denoting $\lambda = \omega^2 l/g$, we find

$$0 = \frac{1}{2}\left(3-\lambda\right)\left(1-\lambda\right) - \frac{1}{2}$$

or

$$\lambda^2 - 4\lambda + 2 = 0 \ .$$

The solution is given by

$$\lambda = 2 \pm \sqrt{2} \ ,$$

and thus

$$\omega = \sqrt{\frac{g}{l}}\sqrt{2 \pm \sqrt{2}} \ .$$

Problem 3: Electromagnetism I

(a) (i) Since the cylinder is infinitely long and the surface potential is independent of z, the system remained invariant under translations in the z direction. Therefore the solution to Laplace's equation is independent of z.

(ii) If $V(r,\varphi)$ is a solution to Laplace's equation, then $-V(r,-\varphi)$ is also a solution. Since $V(R,\varphi) = -V(R,-\varphi) = V_0 \sin\varphi$, both solutions satisfy the same boundary conditions. Since the solution to Laplace's equation is unique for given boundary conditions, it follows that the two solutions are the same, i.e., $V(r,\varphi) = -V(r,-\varphi)$ for all r,φ.

(b) The potential inside the cylinder must be finite at $r = 0$. Thus $b_n = 0$ for all n, including $n = 0$. Since $V(r,\varphi)$ is antisymmetric under $\varphi \to -\varphi$ (see (a)), it follows that $a_0 = 0$ and $\alpha_n = 0$ for all n. Then the potential inside the cylinder takes the simpler form

$$V_{\rm in}(r,\varphi) = \sum_{n=1}^{\infty} a_n r^n \sin n\varphi \ .$$

Since $V_{\rm in}$ satisfies the boundary conditions $V_{\rm in}(R,\varphi) = V_0 \sin\varphi$ at $r = R$, it follows that $a_n = 0$ for $n > 1$ and $a_1 R = V_0$. Thus

$$V_{\rm in}(r,\varphi) = V_0 \left(\frac{r}{R}\right) \sin\varphi \ .$$

(c) The potential outside the cylinder must be well-behaved at $r \to \infty$. This implies that $b_0 = 0$ and $a_n = 0$ for $n \geq 1$. Since $V(r,\varphi)$ is antisymmetric under $\varphi \to -\varphi$, it follows that $a_0 = 0$ and $\beta_n = 0$ for all n. The the potential outside takes the form

$$V_{\rm out}(r,\varphi) = \sum_{n=1}^{\infty} b_n r^{-n} \sin n\varphi \ .$$

Using the boundary conditions at $r = R$, we find that $b_n = 0$ for $n > 1$ and $V_0 = b_1/R$. The potential outside is then

$$V_{\rm out}(r,\varphi) = V_0 \left(\frac{R}{r}\right) \sin\varphi \ .$$

(d) According to Gauss's law, the surface charge density σ at an angle φ is given by

$$(\vec{E}_2 - \vec{E}_1) \cdot \hat{n} = 4\pi\sigma \ ,$$

where $\vec{E}_1$ and $\vec{E}_2$ are the electric fields just inside and just outside the cylinder, and $\hat{n} = \hat{r}$ is the unit vector normal to the surface of the cylinder.

Since $\vec{E} = -\nabla V$ and $\vec{E} \cdot \hat{n} = -\partial V/\partial r|_{r=R}$, the left-hand side of the above equation is simply the discontinuity in $\partial V/\partial r$ across $r = R$, and the surface charge density is given by

$$\sigma = -\frac{1}{4\pi}\left(\frac{\partial V_{\rm out}}{\partial r} - \frac{\partial V_{\rm in}}{\partial r}\right)_{r=R} .$$

Substituting the potentials from parts (b) and (c), we find

$$\sigma = \frac{V_0}{2\pi R}\sin\varphi \ .$$

(e) The local capacitance is defined by $C = dQ/dV$, where $dQ/d\varphi = R\sigma(\varphi)$ is the charge per unit angle (per unit length) and $dV/d\varphi = dV/d\varphi = V_0\cos\varphi$ is the voltage per unit angle. Therefore the local capacitance per unit length of the cylinder is

$$C = \frac{dQ/d\varphi}{dV/d\varphi} = \frac{1}{2\pi}\tan\varphi \ .$$

Problem 4: Electromagnetism II

In the solution below we use Gaussian units.

(a) By symmetry $\vec{E}$ is in the $\hat{r}$ direction. Taking a cylindrical box of radius r and using Gauss's law, we find $\vec{E} = 0$ inside the cylinder and $E(r) = 4\pi\sigma r/R$ outside ($r \geq R$).

(b) By symmetry, $\vec{B}$ is in the z direction. Treating the system as an infinitely long solenoid, $\vec{B} = 0$ outside the cylinder. To find the fieid inside the solenoid we take a rectangular Amperian loop with one side inside the cylinder and the other side outside. Using Ampere's law and a surface current per unit length of $\omega R\sigma$, we find

$$\vec{B} = \frac{4\pi}{c}\omega R\sigma\hat{z} \ .$$

(c) The magnetic field $\vec{B}$ as the same as in part (b). However, ω is now time dependent, and, following Faraday's law of induction, the time dependence of B induces an electric field in the $-\hat{\phi}$ direction. Using a circular Faraday's loop of radius R just inside the cylinder, we find $2\pi RE = -\frac{1}{c}\pi R^2 dB/dt$ implying that

$$\vec{E} = -\frac{4\pi}{2c^2}\alpha R^2\sigma\hat{\phi} \ .$$

(d) Outside the cylinder $\vec{B} = 0$, and $\vec{S} = 0$. Using the results in (b) and (c), we find that just inside the cylinder

$$\vec{S} = -\frac{2\pi}{c^2}\alpha^2 t R^3\sigma^2\hat{r}$$

and the energy flows towards the axis of the cylinder.

(e) The rate energy flow into the cylinder per unit length is

$$\frac{dE_{\text{tot}}}{dt} = 2\pi RS = \frac{4\pi^2}{c^2}\alpha^2 t R^4\sigma^2.$$

The field energy density is given by

$$u = \frac{1}{8\pi}(E^2 + B^2) \ ,$$

and we have to compare the time derivative of $E_{tot} = \int u dV$ to the result above. Since $\vec{E}$ is time-independent, we do not have to calculate its contribution to the energy density. Since B is spatially constant within the cylinder, the total magnetic energy is

$$E_{\text{tot,magnetic}} = \frac{1}{8\pi}\pi R^2\left(\frac{4\pi}{c}\right)2\sigma^2\alpha^2 t^2 R^2 = \frac{2\pi^2}{c^2}\alpha^2 t^2 R^4\sigma^2 \ .$$

Thus

$$\frac{dE_{\mathrm{tot}}}{dt} = \frac{4\pi^2}{c^2}\alpha^2 t R^4 \sigma^2 \ .$$

This is the same as the integral of the Poynting vector calculated above. It expresses the conservation of energy (the rate at which energy is input into the system is equal to the rate of change of the total energy of the system).