QUALIFYING EXAMINATION, Part 2

Solutions

Problem 1: Quantum Mechanics I

(a)

$$\hat{U}(t) = e^{-i\hat{H}t/\hbar} = e^{-i\hat{S}_z\omega t/\hbar} .$$

(b) The normalized state of the system $|t=0\rangle$ can be written by rotating the state $|+\rangle$ by an angle φ about the y axis

$$|t=0\rangle = \hat{R}_y(\varphi)|+\rangle = e^{-i\hat{S}_y\phi/\hbar}|+\rangle$$

Using $\hat{S}_y = (\hbar/2)\sigma_y$ and $\sigma_y^2 = 1$, we find in the spin 1/2 representation

$$\hat{R}_y(\varphi) = \cos\frac{\varphi}{2} - i\sigma_y \sin\frac{\varphi}{2} = \begin{pmatrix} \cos\frac{\varphi}{2} & -\sin\frac{\varphi}{2} \\ \sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{pmatrix} .$$

The state at time $t=0$ is then given by

$$|t=0\rangle = \cos\frac{\varphi}{2}|+\rangle + \sin\frac{\varphi}{2}|-\rangle .$$

(c) At later times, the state $|t\rangle$ is

$$|t\rangle = \cos\frac{\varphi}{2}e^{-i\omega t/2}|+\rangle + \sin\frac{\varphi}{2}e^{i\omega t/2}|-\rangle .$$

The state $|S_x = \hbar/2\rangle$ is obtained from $|+\rangle$ by a rotation of $\pi/2$ about the y axis:

$$|S_x = \hbar/2\rangle = \hat{R}_y(\pi/2)|+\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) .$$

where we have used $R_y(\pi/2) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

$$\begin{aligned} |\langle S_x = +\hbar/2|t\rangle|^2 &= \tfrac{1}{2}\left|\cos\tfrac{\varphi}{2}e^{-i\omega t/2} + \sin\tfrac{\varphi}{2}e^{i\omega t/2}\right|^2 \\ &= \tfrac{1}{2}\left[\cos^2\tfrac{\varphi}{2} + \sin^2\tfrac{\varphi}{2} + \sin\tfrac{\varphi}{2}\cos\tfrac{\varphi}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)\right] \\ &= \tfrac{1}{2}\left[1 + 2\sin\tfrac{\varphi}{2}\cos\tfrac{\varphi}{2}\cos(\omega t)\right] = \tfrac{1}{2}\left[1 + \sin\varphi\cos(\omega t)\right] \end{aligned} .$$

(d) Using the expression in (c) for $|t\rangle$ in terms of $|+\rangle$ and $|-\rangle$, and the fact that in this basis $\hat{S}_x = (\hbar/2)\sigma_x$ (where σ_x is a Pauli matrix), we find

$$\langle t|\hat{S}_x|t\rangle = \frac{\hbar}{2}\cos\frac{\varphi}{2}\sin\frac{\varphi}{2}(e^{-i\omega t} + e^{i\omega t}) = \hbar\cos\frac{\varphi}{2}\sin\frac{\varphi}{2}\cos(\omega t) = \frac{\hbar}{2}\sin\varphi\cos(\omega t)\ .$$

(e) For $\varphi = 0$, the answers to (c) and (d) are, respectively, 1/2 and 0.

Direct calculation: for $\varphi = 0$, $|t = 0\rangle = |+\rangle$ and $|t\rangle = e^{-i\omega t/2}|+\rangle$. Then $|\langle S_x = +\hbar/2|t\rangle|^2 = |\langle S_x = +\hbar/2|+\rangle|^2 = 1/2$ and $\langle t|\hat{S}_x|t\rangle = \langle +|\hat{S}_x|+\rangle = 0$.

Problem 2: Quantum Mechanics II

(a) To evolve from $|\psi_{\text{cold}}\rangle$ at $t=0$ to to $|\psi_{\text{boiled}}\rangle$ at $t=t_B$, we need the unitary matrix element $|U_{2,1}(t_B)|=1$. The smallest positive t_B that satisfy $|\sin \nu t_B|=1$ is

$$t_B = \frac{\pi}{\nu}.$$

(b) We have $U\left(\frac{t_B}{2},0\right)=U\left(\frac{t_B}{2}\right)=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$. At time $t=\frac{t_B}{2}$, $|\frac{t_B}{2}\rangle = U\left(\frac{t_B}{2},0\right)|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$, which collapse to either $|\psi_{\text{cold}}\rangle$ or $|\psi_{\text{boiled}}\rangle$ with probability 1/2 each after we measure σ_z. Then the time evolution $U\left(t_B,\frac{t_B}{2}\right)=U\left(\frac{t_B}{2}\right)$ over the interval $\left[\frac{t_B}{2},\ t_B\right]$ starts from one of these states. The possible outcomes are

1st meas outcome	2nd meas outcome	Probability
"cold"	"cold"	1/4
"cold"	"boiled"	1/4
"boiled"	"cold"	1/4
"boiled"	"boiled"	1/4

(c) We have $U\left(\frac{t_B}{N}\right)=\begin{pmatrix}\cos\frac{\pi}{2N} & -\sin\frac{\pi}{2N}\\ \sin\frac{\pi}{2N} & \cos\frac{\pi}{2N}\end{pmatrix}$. Given the initial state $|t=0\rangle = |\psi_{\text{cold}}\rangle$, the system evolves into $\left|\frac{t_B}{N}\right\rangle = U\left(\frac{t_B}{N}\right)|t=0\rangle = \begin{pmatrix}\cos\frac{\pi}{2N}\\ \sin\frac{\pi}{2N}\end{pmatrix}$ at time $t=\frac{t_B}{N}$ with probability $p_1=\cos^2\frac{\pi}{2N}$ for the first measurement being "cold". Given that the k-th measurement outcome is "cold" with the system collapsing to $|\psi_{\text{cold}}\rangle$ at $t=kt_B/N$, the system evolves into $U\left(\frac{t_B}{N}\right)|\psi_{\text{cold}}\rangle = \begin{pmatrix}\cos\frac{\pi}{2N}\\ \sin\frac{\pi}{2N}\end{pmatrix}$ right before the next measurement, giving the probability $p_{k+1}=\cos^2\frac{\pi}{2N}$ for the $(k+1)$ measurement outcome being "cold." Therefore, the probability of all N measurements yielding the "cold" state is

$$P_{Zeno,N}=p_1p_2\cdots p_N=\cos^{2N}\left(\frac{\pi}{2N}\right).$$

(d) We expand $P_{Zeno,N}=\left(1-\frac{1}{2}\left(\frac{\pi}{2N}\right)^2+o\left(\frac{1}{N^4}\right)\right)^{2N}=1-\frac{\pi^2}{4N}+o\left(\frac{1}{N^2}\right)\approx 1-\frac{\pi^2}{4N}$ to leading order in $1/N$ for $N\gg 1$.

(e) Substituting $N=\frac{t_B}{\Delta t}$ into the expression in (d), we find

$$P_{Zeno,N}=1-\frac{\pi^2\Delta t}{4t_B}+o\left((\Delta t)^2\right).$$

Thus $P_{Zeno,N}\to 1$ when $\Delta t\to 0$. This means that if we continuously observe the pot, we will always observe it in the "cold" state.

Statistical Mechanics I

(a) A zipper with m unzipped links has energy $m\epsilon$ and degeneracy g^m. In the limit $N \to \infty$

$$Z = \Sigma_{m=0}^{\infty} g^m e^{-m\epsilon/(k_B T)} = \frac{1}{1 - g e^{-\epsilon/(k_B T)}} \ ,$$

where k_B is the Boltzmann constant and we used the hint with $x = g e^{-\epsilon/(k_B T)}$ to sum the geometric series.

The series converges for $x < 1$, i.e., for $T < T_M = \frac{\epsilon}{k_B \ln g}$.

(b) Denoting $\beta = \frac{1}{k_B T}$, the average number of unzipped links is

$$\langle m \rangle = \sum_{m=0}^{\infty} m g^m \frac{e^{-\beta m \epsilon}}{Z} = -\frac{1}{Z} \frac{\partial}{\partial(\beta\epsilon)} \sum_{m=0}^{\infty} g^m e^{-\beta m \epsilon} = -\frac{\partial}{\partial(\beta\epsilon)} \ln Z \ ,$$

Using $\ln Z = -\ln(1 - g e^{-\beta\epsilon})$ from (a), we find

$$\langle m \rangle = \frac{g e^{-\beta\epsilon}}{1 - g e^{-\beta\epsilon}} = \frac{1}{g^{-1} e^{\epsilon/(k_B T)} - 1} \ .$$

(c) The probability that zero links are unzipped is

$$P_0 = \frac{1}{Z} = 1 - g e^{-\epsilon/(k_B T)}.$$

Therefore, the probability that one or more links are unzipped is

$$Q = 1 - P_0 = g e^{-\frac{\epsilon}{k_B T}} \ .$$

(d) A state with m unzipped links on the left and m' unzipped links on the right has an energy of $(m + m')\epsilon$ and degeneracy of $g^{m+m'}$. The partition function of the new zipper is thus

$$Z_2 = \sum_{m,m'} g^{m+m'} e^{-(m+m')\epsilon/(k_B T)} = Z^2 = \frac{1}{\left(1 - g e^{-\frac{\epsilon}{k_B T}}\right)^2} \ .$$

This result can also be inferred by arguing that the two ends of the zipper are independent and the same.

Following a similar argument

$$\langle m_2 \rangle = 2\langle m \rangle = 2\frac{g e^{-\frac{\epsilon}{k_B T}}}{1 - g e^{-\frac{\epsilon}{k_B T}}} = 2\frac{1}{g^{-1} e^{\epsilon/(k_B T)} - 1} \ .$$

Problem 4: Statistical Mechanics II

(a) To determine the single-particle density of states, we consider the number of possible single-particle states in a volume $d^d\vec{p}$ of momentum space. Imposing periodic boundary conditions, the allowed values of $\vec{p}$ are $\vec{p} = \frac{2\pi\hbar}{L}\vec{n}$ with $\vec{n}$ having integer components (in d dimensions). Thus the number of allowed momentum values between $\vec{p}$ and $\vec{p} + d\vec{p}$ are

$$L^d \frac{d^d\vec{p}}{(2\pi\hbar)^d} = \frac{VS_d}{(2\pi\hbar)^d} p^{d-1} dp = \frac{VS_d}{(2\pi\hbar)^d} p^{d-1} \frac{dp}{d\epsilon} d\epsilon = \frac{VS_d}{a^{d/s} s (2\pi\hbar)^d} \epsilon^{d/s-1} d\epsilon \; ,$$

where in the last equality we have used $\epsilon = ap^s$ and $d\epsilon = asp^{s-1}dp$.

Thus the single-particle density of state versus energy is

$$g(\epsilon) = \frac{VS_d}{a^{d/s} s (2\pi\hbar)^d} \epsilon^{d/s-1} \; .$$

(b) The grand canonical potential for a non-interacting bose gas is

$$\Omega = -k_B T \ln Z_{\rm GC} = k_B T \sum_{\vec{p}} \ln[1 - e^{-\beta(\epsilon_p - \mu)}] \; ,$$

where the allowed values of $\vec{p}$ are $\vec{p} = \frac{2\pi\hbar}{L}\vec{n}$ with $\vec{n}$ having integer components.

The average particle number is then

$$N = -\frac{\partial \Omega}{\partial \mu} = \sum_{\vec{p}} \frac{1}{e^{\beta(\epsilon_p - \mu)} - 1} \; ,$$

which expresses the relation between N and μ.

At large volume, replacing the sum by an integral gives

$$N \approx V \int \frac{d^d\vec{p}}{(2\pi\hbar)^d} \frac{1}{e^{\beta(\epsilon_p - \mu)} - 1} = \int_0^\infty d\epsilon \, g(\epsilon) \frac{1}{e^{\beta(\epsilon - \mu)} - 1} \; .$$

Changing variables to the dimensionless quantity $x = \beta\epsilon = \beta a p^s$, we have $dx = \beta d\epsilon = \beta a s p^{s-1} dp$, giving

$$N \approx \frac{VS_d}{(\beta a)^{d/s} s (2\pi\hbar)^d} \int_0^\infty dx \frac{x^{d/s-1}}{z^{-1}e^x - 1} = \frac{VS_d}{(\beta a)^{d/s} s (2\pi\hbar)^d} \Gamma(d/s) F_{d/s}(z) \; ,$$

where we have defined the fugacity z by $z = e^{\beta\mu}$ and used the given formula for the integral.

(c) Bose-Einstein condensation occurs when there is a sizeable occupancy of the single-particle ground state (i.e., $\langle n_0 \rangle / N$ is finite), in which case the integral in (b) only gives the contribution N_e of bosons in the excited single-particle states. Bose-Einstein condensation then occurs if the integral remains finite at its maximum, which occurs for $\mu \to \epsilon_0 = 0$ or $z = 1$. Following the information given on the convergence of $F_r(1)$, this occurs for

$$r = d/s > 1 \ .$$

This condition is satisfied by $d = 3, s = 2$.

(d) The critical temperature T_c is determined by the equation found for N in part (b) with $z = 1$ (i.e., $\mu = 0$), since lowering the temperature further will move a finite fraction of particles into the single-particle ground state $\vec{p} = 0$. The corresponding equation is

$$\left(\frac{k_B T_c}{a}\right)^{d/s} \frac{V S_d}{s(2\pi\hbar)^d} \Gamma(d/s) F_{d/s}(1) = N \ .$$

The occupancy of the single-particle ground state $\vec{p} = 0$ for $T < T_c$ is $\langle n_0 \rangle = N - N_e$, where N_e is the number of boson in the excited single-particle states. N_e is given by the r.h.s. of the formula for N in part (b) with $z = 1$

$$N_e = \left(\frac{k_B T}{a}\right)^{d/s} \frac{V S_d}{s(2\pi\hbar)^d} \Gamma(d/s) F_{d/s}(1) \ .$$

Using the above formulas for N_e and N, the constants cancel and we find

$$\frac{\langle n_0 \rangle}{N} = 1 - \frac{N_e}{N} = 1 - \left(\frac{T}{T_c}\right)^{d/s} \ .$$