QUALIFYING EXAMINATION, Part 1

Solutions

Problem 1: Mathematical Methods

(a) Under an infinitesimal transformation a vector shifts as $\vec{v}+d\vec{v} = U(d\theta)\vec{v}$ or $\frac{d\vec{v}}{d\theta} = -\tau\vec{v}$. This has the solution $\vec{v}(\theta) = e^{-\tau\theta}\vec{v}$. Expanding out the exponential gives

$$\begin{aligned} U(\theta) &= \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix}^n = \begin{pmatrix} 1 + \frac{\theta^2}{2!} + \ldots & \theta + \frac{\theta^3}{3!} + \ldots \\ \theta + \frac{\theta^3}{3!} + \ldots & 1 + \frac{\theta^2}{2!} + \ldots \end{pmatrix} \\ &= \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \end{aligned}$$

(b) A group must satisfy closure, associativity, there should be an identity, and every element should have an inverse. The integers satisfy these conditions under addition with 0 being the identity element and the inverse of n is $-n$.

The integers are not a group under multiplication because there is no inverse.

(c) We write the integral as

$$\int_0^\infty \frac{1-\cos 2x}{1+x^2} dx = \frac{1}{2}\Re \int_{-\infty}^{\infty} \frac{1-e^{2ix}}{(x+i)(x-i)}$$

and close the contour with a semicircle at infinity in the upper half plane, picking out the residue of the pole at $x = i$. This gives

$$\frac{1}{2}\Re\left[2\pi i \left(\frac{1-e^{2ix}}{x+i}\right)\bigg|_{x=i}\right] = \frac{\pi}{2}\left(1-e^{-2}\right) .$$

(d) We start with the first polynomial $p_1 = c_1 u_0 = c_1$, which has norm $\langle p_1|p_1\rangle = c_1^2 \int_0^\infty x e^{-x} dx = c_1^2$, and thus $c_1 = 1$.

An orthogonal polynomial will be given by

$$p_2 = c_2(x - \int_0^\infty x^2 e^{-x} dx) = c_2(x-2) ,$$

and it has norm

$$\langle p_2|p_2\rangle = c_2^2 \int_0^\infty x e^{-x}(x-2)^2 dx = c_2^2(3! - 4\cdot 2! + 4\cdot 1) = c_2^2(6-8+4) = 2c_2^2 .$$

Therefore $c_2 = \frac{1}{\sqrt{2}}$, and the two lowest-order normalized orthogonal polynomials are

$$p_1 = 1, p_2 = \frac{x-2}{\sqrt{2}} .$$

Problem 2: Classical Mechanics

(a) The horizontal position of the mass is $x_M = x + L\sin\theta$ and the vertical position is $y_M = -L\cos\theta$. The Lagrangian is then

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(4M)\dot{x}^2 + \frac{1}{2}M(\dot{x}_M^2 + \dot{y}_M^2) - Mg(-L\cos(\theta)) \\ &= \frac{1}{2}(4M)\dot{x}^2 + \frac{1}{2}M(L\dot{\theta}\cos\theta + \dot{x})^2 + \frac{1}{2}M(L\dot{\theta}\sin\theta)^2 + MgL\cos\theta \\ &= \frac{5}{2}M\dot{x}^2 + \frac{1}{2}ML^2\dot{\theta}^2 + ML\dot{x}\dot{\theta}\cos\theta + MgL\cos\theta\ .\end{aligned}$$

(b) In the small oscillation approximation around $\theta = 0$, we have

$$\begin{aligned}\mathcal{L} &\approx \frac{1}{2}\begin{pmatrix}\dot{x} & \dot{\theta}\end{pmatrix}\begin{pmatrix}5M & ML \\ ML & ML^2\end{pmatrix}\begin{pmatrix}\dot{x} \\ \dot{\theta}\end{pmatrix} - \frac{1}{2}MgL\theta^2 \\ &= \frac{1}{2}\begin{pmatrix}\dot{x} & \dot{\theta}\end{pmatrix}\mathcal{M}\begin{pmatrix}\dot{x} \\ \dot{\theta}\end{pmatrix} - \frac{1}{2}\begin{pmatrix}x & \theta\end{pmatrix}\mathcal{V}\begin{pmatrix}x \\ \theta\end{pmatrix}\ ,\end{aligned}$$

where the mass matrix $\mathcal{M}$ and the potential matrix $\mathcal{V}$ are given by

$$\mathcal{M} = \begin{pmatrix}5M & ML \\ ML & ML^2\end{pmatrix}\ , \quad \mathcal{V} = \begin{pmatrix}0 & 0 \\ 0 & MgL\end{pmatrix}\ .$$

The equations of motion are

$$\begin{pmatrix}5M & ML \\ ML & ML^2\end{pmatrix}\begin{pmatrix}\ddot{x} \\ \ddot{\theta}\end{pmatrix} = -\begin{pmatrix}0 & 0 \\ 0 & MgL\end{pmatrix}\begin{pmatrix}x \\ \theta\end{pmatrix}\ .$$

(c) For $\theta = 0$, we find from the equations of motion $\ddot{x} = 0$, whose general solution is

$$x(t) = x_0 + v_0 t\ .$$

The motion has a constant velocity v_0.

(d) The normal frequencies ω are found from the roots of

$$\det(\mathcal{V} - \omega^2\mathcal{M}) = 0\ ,$$

or

$$\det\begin{pmatrix}-5M\omega^2 & -ML\omega^2 \\ -ML\omega^2 & MgL - ML^2\omega^2\end{pmatrix} = 0\ .$$

We find

$$5\omega^2(L^2\omega^2 - gL) - L^2\omega^4 = 0 \ ,$$

or

$$\omega^2(4\omega^2 - 5\frac{g}{L}) = 0 \ .$$

Thus, the two normal frequencies are

$$\omega = 0, \sqrt{\frac{5g}{4L}}.$$

The normal mode vectors $\vec{a}$ are the solutions of

$$(\mathcal{V} - \omega^2\mathcal{M})\vec{a} = 0 \ ,$$

or

$$\begin{pmatrix} -5\omega^2 & -L\omega^2 \\ -L\omega^2 & gL - L^2\omega^2 \end{pmatrix} \vec{a} \ = \ 0 \ .$$

We can also find the normal modes by assuming a solution of the form

$$\begin{pmatrix} x(t) \\ \theta(t) \end{pmatrix} = Re(\vec{a}e^{i\omega t}) \ .$$

(i) $\omega = 0$

$$\begin{pmatrix} 0 & 0 \\ 0 & -gL \end{pmatrix} \begin{pmatrix} a_x \\ a_\theta \end{pmatrix} \ = \ 0,$$

so $a_\theta = 0$ and this solution simply corresponds to

$$\vec{a} \ = \ \begin{pmatrix} x_0 \\ 0 \end{pmatrix}.$$

(ii) $\omega = \sqrt{\frac{5g}{4L}}$

$$\begin{pmatrix} \frac{25}{4L} & \frac{5}{4} \\ \frac{5}{4} & \frac{L}{4} \end{pmatrix} \begin{pmatrix} a_x \\ a_\theta \end{pmatrix} \ = \ 0,$$

so $5a_x + La_\theta = 0$ and the solution is

$$\vec{a} \ = \ c\begin{pmatrix} 1 \\ -\frac{5}{L} \end{pmatrix}.$$

The general solution is a superposition of the two normal modes

$$x(t) = x_0 + v_0 t + A\cos\left(\sqrt{\frac{5g}{4L}}t - \delta\right)$$

$$\theta(t) = -\frac{5}{L}A\cos\left(\sqrt{\frac{5g}{4L}}t - \delta\right) \ .$$

There are 4 undetermined real parameters: x_0, v_0, A and δ.

Problem 3: Electromagnetism I

In the solution below we use SI units.

(a)

$$\vec{E} = \frac{q\hat{r}}{4\pi\epsilon_0 r^2} \ .$$

(b) We apply Gauss's Law to a small pillbox across the surface of charge. The electric field is perpendicular to the charged surface by symmetry arguments, so the electric flux is

$$2AE = \sigma A/\epsilon_0 \ .$$

We obtain

$$E = \frac{1}{2\epsilon_0}\sigma \ .$$

(c) The surface charge density of the disk is $\sigma = Q/(\pi R^2)$. We calculate the potential on the symmetry axis for rings of radius ρ and width $d\rho$ and integrate ρ from 0 to R. The distance to the point of observation on the z axis is $\sqrt{\rho^2 + z^2}$. We find

$$V(z) = \int_0^R \frac{\sigma}{4\pi\epsilon_0} \frac{2\pi\rho\,\mathrm{d}\rho}{\sqrt{\rho^2 + z^2}} = \frac{\sigma}{2\epsilon_0}\left.\sqrt{\rho^2 + z^2}\right|_0^R = \frac{\sigma}{2\epsilon_0}\left(\sqrt{z^2 + R^2} - z\right) .$$

The electric field is given by

$$E(z) = -\frac{\mathrm{d}V}{\mathrm{d}z} = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2 + R^2}}\right)$$

(d) When $z \to 0$, $E \to \frac{\sigma}{2\epsilon_0}$, just like the field of an infinite sheet of charge.

When $z \to \infty$, we have

$$E = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{1}{\sqrt{1 + R^2/z^2}}\right) \sim \frac{\sigma}{2\epsilon_0}\left(1 - 1 + \frac{1}{2}\frac{R^2}{z^2}\right) = \frac{\sigma\pi R^2}{4\pi\epsilon_0 z^2} = \frac{Q}{4\pi\epsilon_0 z^2},$$

the field of a point charge Q on the axis.

(e) Because of axial symmetry, the potential is independent of the azimuthal angle φ and $V = V(r, \theta)$. The general solution of Laplace's equation for axial symmetry in spherical coordinates is

$$V(r, \theta) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta) \ .$$

For $r \to \infty$, $V \to 0$ and therefore all $A_l = 0$. To determine B_l, we use as boundary condition the potential $V(r, \theta = 0)$ calculated in part (c) along the symmetry axis $\theta = 0$ for $z = r$. We have

$$V(r, \theta = 0) = \frac{\sigma}{2\epsilon_0}\left[r\left(1 + R^2/r^2\right)^{1/2} - r\right] = \sum_l \frac{B_l}{r^{l+1}} \ ,$$

where we have used $P_l(1) = 1$. Using the binomial expansion

$$(1 + x)^\alpha = \sum_{k=0}^{\infty}\left(\begin{array}{c}\alpha \\ k\end{array}\right) x^k = 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!}x^2 + \frac{\alpha(\alpha - 1)(\alpha - 2)}{3!}x^3 + \dots$$

for $x = (R/r)^2$, we find

$$\frac{\sigma}{2\epsilon_0}\sum_{k=1}^{\infty}\binom{1/2}{k}\frac{R^{2k}}{r^{2k-1}} = \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}} \ .$$

The above equation determines the coefficients B_l. The final result is

$$V(r, \theta) = \frac{\sigma}{2\epsilon_0}\sum_{k=1}^{\infty}\binom{1/2}{k}\frac{R^{2k}}{r^{2k-1}}P_{2k-2}(\cos\theta) \ .$$

(f) For $z \gg R, a$, the plane of disks looks like a plane with a uniform charge density $\sigma = Q/a^2$. The field at large distances is then

$$E = \frac{1}{2\epsilon_0}\frac{Q}{a^2}$$

in the direction perpendicular to the plane.

Problem 4: Electromagnetism II

In the solution below we use SI units.

(a)

$f = \omega/2\pi$, $\lambda = c/f$ or $2\pi/k$; $\vec{B} = \frac{E_0}{c}\hat{z}\cos(kx - \omega t)$.

(b)

$\vec{E}, \vec{B}$ are perpendicular and in phase, and $B = E/c$ for an electromagnetic wave. The time average of $\cos^2(\omega t)$ is $1/2$, so that

$$\langle \vec{S} \rangle = \frac{E_0^2}{2c\mu_0} \ .$$

(c) The force exerted on the disk in the direction normal to its surface is given by the total momentum per second transferred by the light in this direction. The momentum transferred per unit area per second is $p_n c \cos\theta$ where $p_n = p\cos\theta$ is the component of the momentum density of the wave along the normal to the disk surface. The total momentum transfer per second to the absorbing disk is then

$$(p_n c \cos\theta)(\pi r^2) = (pc\cos^2\theta)(\pi r^2) = (\frac{1}{c}S\cos^2\theta)(\pi r^2) = \frac{1}{2c^2\mu_0}E_0^2\pi r^2\cos^2\theta \ .$$

For the reflecting surface, the corresponding total momentum transfer is twice as much, and thus the force is twice as large.

(d) The torque is given by the sum of $\vec{r} \times \vec{F}$ for all forces in the problem. The reflecting mirror receives twice the force so the net torque is

$$\tau = \frac{1}{2c^2\mu_0}E_0^2\pi r^2 R\cos^2\theta$$

in the $\hat{z}$ direction.

Taking an average over one full rotation in θ, $\langle\cos^2\theta\rangle = 1/2$, and the average torque is

$$\langle\tau\rangle = \frac{1}{4c^2\mu_0}E_0^2\pi r^2 R \ .$$

(e)

$\tau = I\ddot{\theta}$, where I is the moment of inertia of the rod plus disks. In the limit $R \gg r$, we have $I = 2mR^2$.

We find

$$\langle\ddot{\theta}\rangle = \frac{1}{8mRc^2\mu_0}E_0^2\pi r^2 \ .$$