

QUALIFYING EXAMINATION, Part 2

Solutions

Problem 1: Quantum Mechanics I

(a) The even-parity solutions are of the form $\cos(kx)$ and the odd-parity solutions have the form $\sin(kx)$. The boundary conditions for the stationary wave function are $\psi(\pm a) = 0$ and lead to the quantization of k . The normalized even- and odd-parity solutions are then given by

$$\begin{aligned}\psi_{n \text{ odd}}(x) &= \frac{1}{\sqrt{a}} \cos(k_n x) = \frac{1}{\sqrt{a}} \cos\left(n \frac{\pi x}{2a}\right) \\ \psi_{n \text{ even}}(x) &= \frac{1}{\sqrt{a}} \sin(k_n x) = \frac{1}{\sqrt{a}} \sin\left(n \frac{\pi x}{2a}\right).\end{aligned}$$

The corresponding energy eigenvalues are

$$E_n = \frac{(\hbar k_n)^2}{2m} = \frac{\hbar^2 \pi^2 n^2}{8ma^2}.$$

(b) In first-order perturbation theory

$$\Delta E_n^{(1)} = \langle \psi_n | H' | \psi_n \rangle = \langle \psi_n | \beta E_1 a \delta(x) | \psi_n \rangle = \beta E_1 a \psi_n^2(0).$$

We find

$$\begin{aligned}\Delta E_{n \text{ odd}}^{(1)} &= \beta E_1 \quad (\text{even parity}) \\ \Delta E_{n \text{ even}}^{(1)} &= 0 \quad (\text{odd parity}).\end{aligned}$$

In second-order perturbation theory

$$\Delta E_n^{(2)} = \sum_{l \neq n} \frac{|\langle \psi_l | H' | \psi_n \rangle|^2}{E_n - E_l} = \beta^2 E_1^2 a^2 \sum_{l \neq n} \frac{\psi_l^2(0) \psi_n^2(0)}{E_n - E_l}.$$

For odd values of n , we obtain

$$\Delta E_{n \text{ odd}}^{(2)} = \sum_{l \text{ odd} \neq n} \frac{\beta^2 E_1^2}{E_1(n^2 - l^2)} = -\frac{\beta^2 E_1}{4n^2},$$

where in the last equality we use the formula given in the problem.

For even values of n

$$\Delta E_{n \text{ even}}^{(2)} = 0.$$

(c) We use the piecewise solution for the wave function

$$\psi(x) = \begin{cases} A_+ \sin(k(x-a)) & \text{for } 0 < x < a \\ A_- \sin(k(x+a)) & \text{for } -a < x < 0 \end{cases}$$

There are two matching conditions at $x = 0$:

(i) The continuity of the wave function

$$\psi(0^+) = \psi(0^-) \Rightarrow -A_+ \sin(ka) = A_- \sin(ka)$$

(ii) The discontinuity of $\psi'(x)$ at $x = 0$ is determined by integrating the Schrödinger equation in a small interval around $x = 0$. We find

$$\psi'(0^+) - \psi'(0^-) = \frac{2m}{\hbar^2} \beta E_1 a \psi(0)$$

or

$$A_+ k \cos(ka) - A_- k \cos(ka) = A_- \frac{2m}{\hbar^2} \beta E_1 a \sin(ka) .$$

There are two classes of solutions:

I.

$$\sin(ka) = 0 \quad \text{and} \quad A_+ = A_-$$

Then $k_n = n \frac{\pi}{2a}$ with n even, and the corresponding energy eigenvalues are

$$E_n = \frac{\hbar^2 \pi^2 n^2}{8ma^2} .$$

These eigenvalues are the same as the unperturbed problem (i.e., without the δ function potential).

II. If $\sin(ka) \neq 0$, then from the first matching condition we have $A_+ = -A_-$. The second condition leads to

$$\cot(ka) = -\frac{m\beta E_1 a^2}{(ka)\hbar^2} .$$

This is an implicit equation that determines the allowed discrete values of k . The energy eigenvalues are then given by $E = \hbar^2 k^2 / 2m$.

(d) In class I of solutions, the energy eigenvalues are the same as the unperturbed problem, in agreement with first-order perturbation theory.

For class II of solutions in the limit $\beta \rightarrow 0$, we have $\cot(ka) = 0$ or $k_n = n \frac{\pi}{2a}$ with n odd. Solving the implicit equation for k to first-order in β we find

$$k_n a \simeq n\pi/2 + \frac{m\beta E_1 a^2}{(n\pi/2)\hbar^2} + \dots$$

The corresponding energy eigenvalues are then given by

$$\begin{aligned} E_n &\simeq \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2}{2ma^2} \left[n\pi/2 + \frac{m\beta E_1 a^2}{(n\pi/2)\hbar^2} \right]^2 \\ &= \frac{\hbar^2 \pi^2 n^2}{8ma^2} + \beta E_1 + \dots \end{aligned}$$

where the last equality holds to first order in β . This is the same as $\Delta E_{n \text{ odd}}^{(1)}$ in part (b).

Problem 2: Quantum Mechanics II

(a)

$$\hat{H}_{Kerr} |n\rangle = \frac{\hbar\chi}{2} n(n-1) |n\rangle .$$

(b)

$$|\psi(t)\rangle = e^{-i\hat{H}_{Kerr}t/\hbar} |\alpha\rangle = e^{-|\alpha^2/2|} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-i\frac{\chi t}{2}n(n-1)} |n\rangle .$$

$$\langle\alpha|\psi(t)\rangle = e^{-|\alpha^2|} \sum_{n=0}^{\infty} \frac{|\alpha^2|^n}{n!} e^{-i\frac{\chi t}{2}n(n-1)} .$$

(c) For $\chi t = 2\pi m$ with integer m , we have

$$\langle\alpha|\psi\left(t = \frac{2\pi m}{\chi}\right)\rangle = e^{-|\alpha^2|} \sum_{n=0}^{\infty} \frac{|\alpha^2|^n}{n!} e^{-im\pi n(n-1)} = e^{-|\alpha^2|} \sum_{n=0}^{\infty} \frac{|\alpha^2|^n}{n!} = 1 .$$

Since there are no other solutions, the smallest positive solution is $t_{revival} = \frac{2\pi}{\chi}$.

(d) At time $t = \frac{t_{revival}}{2}$

$$\begin{aligned} \left|\psi\left(\frac{t_{revival}}{2}\right)\right\rangle &= e^{-|\alpha^2/2|} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-i\frac{\pi}{2}n(n-1)} \\ &= e^{-|\alpha^2/2|} \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{\sqrt{n!}} e^{-i\frac{\pi}{2}n^2} \\ &= e^{-|\alpha^2/2|} \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{\sqrt{n!}} \frac{e^{-i\pi/4} + e^{i\pi/4}(-1)^n}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2}} (e^{-i\pi/4} |i\alpha\rangle + e^{i\pi/4} |-i\alpha\rangle) , \end{aligned}$$

where we have used the hint in the second and third equalities. The state at $t_{revival}/2$ is thus a superposition of two coherent states $|i\alpha\rangle$ and $|-i\alpha\rangle$, with coefficients $\frac{e^{\mp i\pi/4}}{\sqrt{2}}$, respectively.

Statistical Mechanics I

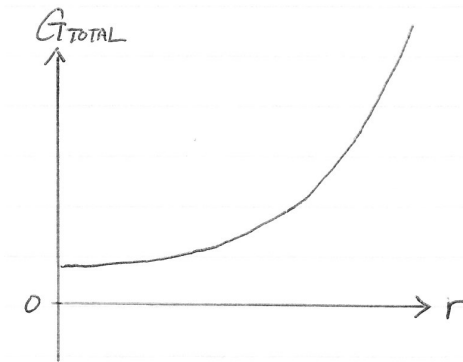
(a)

$$G = \mu_l N_l + \mu_v (N - N_l) = N\mu_v + N_l(\mu_l - \mu_v) = N\mu_v + \frac{4\pi r^3}{3v_l}(\mu_l - \mu_v).$$

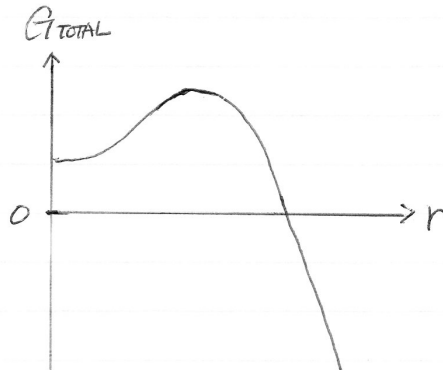
(b)

$$G_{\text{total}} = N\mu_v + \frac{4\pi r^3}{3v_l}(\mu_l - \mu_v) + 4\pi\sigma r^2.$$

(c) For $\mu_l - \mu_v > 0$, G_{tot} has a global minimum at $r = 0$ (see figure below). Thus, starting from any finite radius, the drop will spontaneously decrease its size and evaporate.



For $\mu_l - \mu_v < 0$, G_{tot} has a local minimum at $r = 0$ and a local maximum at a certain critical value of r (see figure below). Thus the drop will either shrink to radius zero or increase in size, depending upon whether or not its initial radius is smaller than this critical radius.



(d) We saw in part (c) that for $\mu_l - \mu_v < 0$, the drop either evaporates completely or grows. The critical radius r_c that separates these two types of behavior can be determined from the extremum of G_{tot} :

$$\left. \frac{dG_{tot}}{dr} \right|_{r=r_c} = -\frac{4\pi r_c^2}{v_l}(\mu_v - \mu_l) + 8\pi\sigma r_c = 0 ,$$

or

$$r_c = \frac{2\sigma v_l}{\mu_v - \mu_l} .$$

(e) If p_0 is the pressure of a vapor in equilibrium with a large flat surface of water, then the reference chemical potential is just the chemical potential of the liquid, i.e., $\mu_v^0 = \mu_l$.

Therefore, using the ideal gas expression for μ_v given in the problem $\mu_v - \mu_l = k_B T \ln(p/p^0)$ and

$$r_c = \frac{2\sigma v_l}{k_B T \ln(p/p^0)} .$$

The relative humidity is then given by

$$\frac{p}{p^0} = \exp\left(\frac{2\sigma v_l}{k_B T r_c}\right) .$$

Drops will grow more easily when their critical radius r_c is smaller, or according to the formula above, when the relative humidity p/p^0 is higher.

Problem 4: Statistical Mechanics II

(a) Using, e.g., plane wave solutions with periodic boundary conditions $e^{ikL} = 1$, the single-particle density of states in momentum space is $D(k) = 2\frac{L}{2\pi}$ where the factor of 2 counts for the spin degeneracy. The single-particle density of states vs. energy is determined by

$$D(\epsilon)d\epsilon = 2D(k)dk = 2L/\pi ,$$

where the factor of 2 accounts for $+k$ and $-k$. A similar result can be found by using hard-wall boundary conditions at the edges. Using the non-relativistic dispersion relation

$$\epsilon = \frac{\hbar^2 k^2}{2m}$$

we find

$$D(\epsilon) = \frac{L}{\pi} \left(\frac{2m}{\hbar^2} \right)^{1/2} \epsilon^{-1/2} .$$

(b) The Fermi energy is determined by the energy of the highest occupied level at $T = 0$, namely

$$\int_0^{\epsilon_F} D(\epsilon)d\epsilon = N .$$

Using the expression for $D(\epsilon)$ found in (a), we obtain

$$\epsilon_F = \pi^2 \frac{\hbar^2}{8m} \left(\frac{N}{L} \right)^2 .$$

(c) The energy of the system at $T = 0$ is given by

$$E = \int_0^{\epsilon_F} \epsilon D(\epsilon)d\epsilon ,$$

while the total number of particles is determined from

$$N = \int_0^{\epsilon_F} D(\epsilon)d\epsilon .$$

Using $D(\epsilon) \propto \epsilon^{-1/2}$, we find the energy per particle

$$\frac{E}{N} = \frac{\int_0^{\epsilon_F} \epsilon^{1/2} d\epsilon}{\int_0^{\epsilon_F} \epsilon^{-1/2} d\epsilon} = \frac{(2/3)\epsilon_F^{3/2}}{2\epsilon_F^{1/2}} = \frac{\epsilon_F}{3} .$$

(d) The chemical potential at temperature T is determined by

$$N = \int_0^{\infty} f(\epsilon)D(\epsilon)d\epsilon .$$

Using the expansion given in the problem, we get

$$N \approx \int_0^\mu D(\epsilon) d\epsilon + \frac{\pi^2}{6} (kT)^2 D'(\epsilon_F),$$

where in the last term on the r.h.s. we took $\mu \approx \epsilon_F$ since this term is already second order in T . We also recall that

$$N = \int_0^{\epsilon_F} D(\epsilon) d\epsilon.$$

Taking the difference of the above two expressions, we get

$$0 = (\mu - \epsilon_F) D(\epsilon_F) + \frac{\pi^2}{6} (kT)^2 D'(\epsilon_F),$$

where we assume that μ is close to ϵ_F . Since $D(\epsilon) \propto \epsilon^{-1/2}$, we obtain

$$\mu = \epsilon_F + \frac{\pi^2 (kT)^2}{12 \epsilon_F}.$$

(e) The above equation indicates that the chemical potential increases with T . This is a result of the decreasing single-particle density of states as a function of energy ϵ . As the temperature increases, more particles occupy states above μ , but given the decreasing density of states, the chemical potential must increase to accommodate this. This is in contrast to the 3D Fermi gas, where the single-particle density of states increases with energy and the chemical potential decreases with increasing temperature.