QUALIFYING EXAMINATION, Part 1

Solutions

Problem 1: Mathematical Methods

(a) Keeping only the lowest power of x needed, we find

$$\begin{aligned}\frac{1}{x^2}-\frac{1}{\sin^2 x} &= \frac{1}{x^2}-\frac{1}{(x-x^3/6\ldots)^2}\\ &= \frac{1}{x^2}\left(1-\frac{1}{1-2x^2/3}\right)\\ &= \frac{1}{x^2}\left[1-(1+2x^2/3!)\right]\\ &= \frac{1}{x^2}\left(\frac{-2x^2}{3!}\right)=-\frac{1}{3}\,.\end{aligned}$$

(b) The sum of finite number of terms is finite. For the rest, the ratio of successive terms test gives the condition

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{\frac{(n+1)!)^2 z^{n+1}}{(2n+2)!}}{\frac{(n!)^2 z^n}{(2n)!}}\right| &< 1\\ \lim_{n\to\infty}\left|\frac{(n+1)^2 z}{(2n+2)(2n+1)}\right| &< 1\\ |z| &< 4\,.\end{aligned}$$

(c) We expand

$$y=\sum_m c_m x^m\,.$$

Then

$$\begin{aligned}y'' &= \sum_m c_m x^{m-2} m(m-1)\\ &= \sum_m c_{m+2} x^m (m+2)(m+1)\quad \text{(dummy label } m\to m+2\text{)}\\ x^2 y'' &= \sum_m c_m x^2 m(m-1)\\ -2xy' &= -2\sum_m c_m x^m m\\ 2y &= 2\sum_m c_m x^m\,.\end{aligned}$$

Collecting the coefficient of x^m and setting it to 0, we find

$$\frac{c_{m+2}}{c_m} = -\frac{(m-1)(m-2)}{(m+1)(m+2)} \; .$$

Choosing c_0 and c_1, arbitrarily, we find $c_2 = -1$ and all others 0. Thus the general solution is given by

$$y = c_0(1 - x^2) + c_1 x \; .$$

(d)

$$\begin{aligned}
\int_0^\infty \frac{x \sin x}{x^2 + a^2} dx &= \frac{1}{2} \int_{-\infty}^\infty \frac{x \sin x}{x^2 + a^2} dx \quad \text{(even integrand)} \\
&= \Im \left(\frac{1}{2} \int_{-\infty}^\infty \frac{z e^{iz}}{z^2 + a^2} dz \right) \quad (\Im \text{ is imaginary part}) \\
&= \Im \left[\frac{1}{2} 2\pi i \frac{i a e^{-a}}{2 i a} \right] \quad \text{(close contour in UHP, pick up residue at } z = ia) \\
&= \frac{\pi}{2} e^{-a} \; .
\end{aligned}$$

Problem 2: Classical Mechanics

(a) The Hamiltonian is

$$H = \frac{p^2}{2m} - ax^2 + \lambda x^4 \,.$$

Hamilton's equations of motion are given by

$$\dot{p} = -\frac{\partial H}{\partial x} = 2ax - 4\lambda x^3, \qquad \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \,.$$

(b) Equilibrium requires $\dot{p} = 0 = x(2a - 4\lambda x^2)$ so that $x = \pm\sqrt{\frac{a}{2\lambda}}$ and $x = 0$.

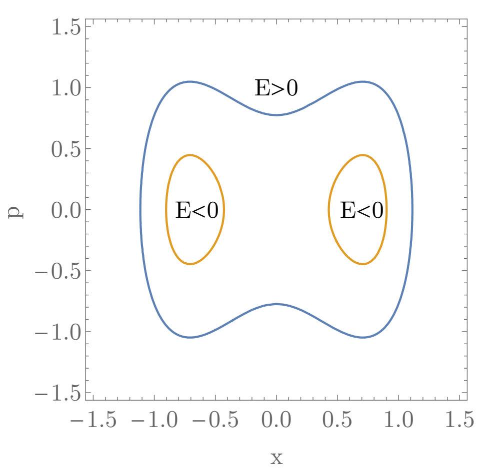
Around $x = 0$ we have $U'' = -2a$ so it is unstable since the second derivative is negative.

Around $x = \pm\sqrt{\frac{a}{2\lambda}}$ we have

$$U(\pm\sqrt{\frac{a}{2\lambda}} + \epsilon) = -a(\pm\sqrt{\frac{a}{2\lambda}} + \epsilon)^2 + \lambda(\pm\sqrt{\frac{a}{2\lambda}} + \epsilon)^4 = -\frac{a^2}{4\lambda} + 2a\epsilon^2 + \ldots$$

Thus, $U'' = 4a > 0$ so the equilibrium points are stable and the effective spring constant for the oscillations is $k_{eff} = 4a$. The oscillation frequency is then $\omega = \sqrt{\frac{k_{eff}}{m}} = \sqrt{\frac{4a}{m}}$.

(c)



(d) The principle of least action is a variational principle which states that the system evolves in a way which extremizes the classical action S.

In the Hamiltonian formulation, the classical action for evolution from t_1 to t_2 is $S = \int_{t_1}^{t_2} (p\dot{x} - H)$ and the principle states that the variation is $\delta S = 0$ under small changes in the configuration $\{x(t) \to x(t) + \delta x(t),\ p(t) \to p(t) + \delta p(t)\}$, where the coordinates are held fixed at the endpoints, $\delta x(t_1) = \delta x(t_2) = \delta p(t_1) = \delta p(t_2) = 0$. (We would also accept a formulation of the principle of least action that leads to Lagrange's equations.)

In the situation asked about in the problem, the classical action is

$$S = \int_0^T (p\dot{x} - H) = -\int_0^T U\left(\pm\sqrt{\frac{a}{2\lambda}}\right) = -T\left(-\frac{a^2}{4\lambda}\right) = \frac{a^2 T}{4\lambda}\ .$$

Problem 3: Electromagnetism I

(a) The trace of the second rank tensor Q is

$$\sum_{i=1}^{3} Q_{ii} = \int \sum_{i} \left(3x_i'^2 - r'^2\right) \rho(x')\, \mathrm{d}^3x' = \int \left(3r'^2 - 3r'^2\right) \rho(x')\, \mathrm{d}^3x' = 0\ .$$

Q_{ij} is real, symmetric, and traceless. Therefore it has 5 independent components.

(b) q and $\mathbf{p}$ are, respectively, the total charge and dipole moment of the charge distribution. They are given by

$$q = \int \rho(\mathbf{x}')\ \mathrm{d}^3x'\ , \quad \mathbf{p} = \int \rho(\mathbf{x}')\mathbf{r}'\ \mathrm{d}^3x'\ .$$

(c)

$$q = e - e + e - e = 0\ .$$

(d)

$$\begin{aligned} p_x &= ea - ea - -ea - ea = 0 \\ p_y &= -ea + ea - ea - -ea = 0 \\ p_z &= 0\ . \end{aligned}$$

Thus $\mathbf{p} = \mathbf{0}$.

(e)

$$\begin{aligned} Q_{xx} &= (3a^2 - 2a^2)(e - e + e - e) = 0 \\ Q_{yy} &= (3a^2 - 2a^2)(-e + e + e - e) = 0 \\ Q_{zz} &= 0 \\ Q_{xy} &= 3a^2(-e) - 3a^2e + 3a^2(-e) - 3a^2e = -12a^2 \\ Q_{xz} &= 0 \\ Q_{yz} &= 0\ . \end{aligned}$$

Thus

$$Q = \begin{pmatrix} 0 & -12ea^2 & 0 \\ -12ea^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\ .$$

(f) Since q and $\mathbf{p}$ are zero, only the quadrupole term survives. We then have

$$\Phi(\mathbf{x}) = 2\frac{1}{2}\left(-12ea^2\frac{xy}{r^5}\right)\ .$$

The electric field is

$$\begin{aligned} \mathbf{E} &= -\nabla\left(-12ea^2\frac{xy}{r^5}\right) \\ &= \left(\frac{12ea^2y}{r^5} - \frac{60ea^2x^2y}{r^7}, \frac{12ea^2x}{r^5} - \frac{60ea^2xy^2}{r^7}, -\frac{60ea^2xyz}{r^7}\right)\ . \end{aligned}$$

Problem 4: Electromagnetism II

(a) Define $k = \omega/c$. The electric and magnetic fields are now given by

$$\vec{E} = E_0 \sin(kz - \omega t)\hat{x} \ ,$$

and

$$\vec{B} = B_0 \sin(kz - \omega t)\hat{y} \, .$$

We can relate E_0 and B_0 by Maxwell's equations; in SI units we have $B_0 = E_0/c$, while in Gaussian units, we have $E_0 = B_0$.

(b) The Poynting vector is

$$\begin{aligned} \vec{S} &= \frac{1}{\mu_0}\vec{E} \times \vec{B} \, (\text{SIunits}) \\ \vec{S} &= \frac{c}{4\pi}\vec{E} \times \vec{B} \, (\text{Gaussianunits}) \end{aligned}$$

Recall that the average of $\sin^2$ over a period is 1/2. Substituting and taking time-averages, we have

$$\begin{aligned} \langle S \rangle &= \frac{E_0^2}{2\mu_0 c} \, (\text{SIunits}) \\ &= \frac{cE_0^2}{8\pi} \, (\text{Gaussianunits}) \end{aligned}$$

(c) The acceleration of the electron is $a = eE/m$; plugging this into the Larmor formula (and time averaging) we get

$$\begin{aligned} \langle P_{\text{rad}} \rangle &= \frac{1}{4\pi\epsilon_0}\frac{1}{3}\frac{e^4 E_0^2}{m_e^2 c^3} \, (\text{SIunits}) \\ &= \frac{1}{3}\frac{e^4 E_0^2}{m_e^2 c^3} \, (\text{Gaussianunits}) \, . \end{aligned}$$

(d) The ratio of the previous two parts is σ. This is simplest in Gaussian units, where we immediately find that

$$\sigma = \frac{8\pi}{3}\frac{e^4}{m_e^2 c^4} = \frac{8\pi}{3} r_e^2 \, .$$

We get the same result in SI units, using $\mu_0 \epsilon_0 = 1/c^2$.

(e) The integral of $d\sigma/d\Omega$ over all angles must equal σ. This implies

$$2\pi A \int_{-1}^{1} \sin^2\theta \, d\cos\theta = \frac{8\pi}{3} r_e^2 \, .$$

Using $\sin^2\theta = 1 - \cos^2\theta$, we have

$$2\pi A \int_{-1}^{1} (1 - \cos^2\theta) d\cos\theta = 2\pi \left(2 - \frac{2}{3}\right) A = \frac{8\pi}{3} r_e^2$$

which yields $A = r_e^2$.

(f) For the x polarized wave $\theta = \pi/2 - \theta$, and so $d\sigma/d\Omega = r_e^2 \cos^2\alpha$.

For the y polarized wave, $\theta = \pi/2$ and we have $d\sigma/d\Omega = r_e^2$.