QUALIFYING EXAMINATION, Part 1

1:00 pm – 5:00 pm, Thursday August 31, 2017

Attempt all parts of all four problems.

Please begin your answer to each problem on a separate sheet, write your 3 digit code and the problem number on each sheet, and then number and staple together the sheets for each problem. Each problem is worth 100 points; partial credit will be given.

Calculators and cell phones may NOT be used.

Problem 1: Mathematical Methods

(a) (20 points) Evaluate

$$\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right) .$$

(b) (20 points) Find the radius of convergence of the series

$$\sum_{n=12017}^{\infty}\frac{(n!)^2 z^n}{(2n!)} .$$

(c) (30 points) Solve by a power series to find the general solution of the differential equation

$$(x^2+1)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0 .$$

(d) (30 points) Use Cauchy's theorem to evaluate

$$\int_0^{\infty}\frac{x\sin x}{x^2+a^2}dx$$

assuming a is real. Justify your choice of the contour integration.

Problem 2: Classical Mechanics

A particle of mass m moves along the x-axis in a potential $U(x) = -ax^2 + \lambda x^4$, where a and λ are positive constants.

(a) (25 points) Write down the Hamiltonian $H(p, x)$ describing this system and find expressions for the time derivatives $\dot{x}$ and $\dot{p}$ using Hamilton's equations of motion.

(b) (35 points) Find the equilibrium values of x, and for each one determine if it is stable or unstable. Find the frequency of small oscillations ω around the stable points.

(c) (20 points) Make a qualitative sketch of a phase space orbit [i.e., a trajectory in the (x, p) plane] for a total energy $E > 0$. How does the plot change for $E < 0$?

Hint: it might be helpful to first make a qualitative sketch of $U(x)$.

(d) (20 points) State Hamilton's principle of least action. Consider a trajectory where the particle sits stationary at a stable equilibrium point for a total time T. What is the classical action S along this trajectory?

Problem 3: Electromagnetism I

The electric potential for a charge distribution $\rho(\mathbf{x}')$ may be computed from

$$\Phi(\mathbf{x}) = \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, \mathrm{d}^3x' \, .$$

At large distances from a localized charge distribution, this may be may be expanded as

$$\Phi(\mathbf{x}) = \frac{q}{r} + \frac{\mathbf{p} \cdot \mathbf{x}}{r^3} + \frac{1}{2} \sum_{i,j} Q_{ij} \frac{x_i x_j}{r^5} + \ldots,$$

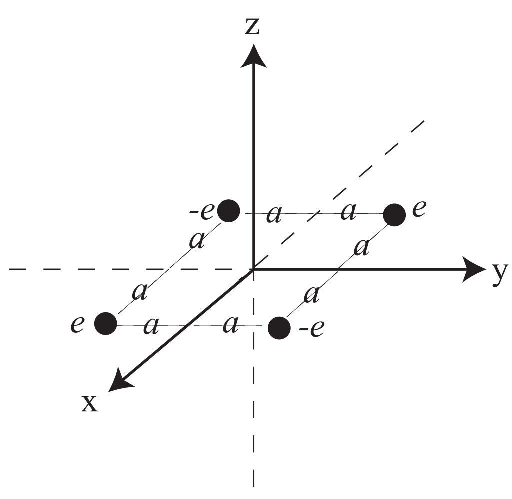
where x_i are the cartesian coordinates of $\mathbf{x}$, $r = |\mathbf{x}|$, and Q_{ij} is the quadrupole moment tensor of the charge distribution

$$Q_{ij} = \int \left(3x'_i x'_j - r'^2 \delta_{ij}\right) \rho(\mathbf{x}') \, \mathrm{d}^3x' \, .$$

(a) (20 points) What is the trace of Q_{ij}? How many independent components does Q_{ij} have? Explain your answer.

(b) (20 points) What are q and $\mathbf{p}$ in the expansion above? Write expressions for q and $\mathbf{p}$ that are analogous to the expression for Q_{ij}.

In the following, assume a distribution of four point charges $\pm e$, each at a distance a from the x and y axes in the $z = 0$ plane as shown in the figure below.



(c) (10 points) What is q for the charge distribution?

(d) (10 points) What is $\mathbf{p}$ for the charge distribution? Explain your answer.

(e) (20 points) Compute Q_{ij} for the charge distribution.

(f) (20 points) What is the electric potential outside the charge distribution due to the combination of the three terms in the expansion above? Compute the corresponding electric field.

Problem 4: Electromagnetism II

You may solve this problem either in SI or in Gaussian units. Make sure to specify which at the start.

A plane electromagnetic wave with angular frequency ω is traveling in the $+z$ direction in the vacuum. Assume its electric field is linearly polarized in the x direction, and that its amplitude is E_0.

(a) (10 points) Write expressions for the electric field $\vec{E}$ and the magnetic field $\vec{B}$ as a function of spatial coordinates and time. Express your answers in terms of E_0, ω and c (the speed of light).

For all the remaining parts of this problem, give your answers in *time-averaged* quantities.

(b) (20 points) Find the power per unit area carried by this wave through a surface normal to the propagation direction.

Hint: the Poynting vector, which describes the energy flux density (i.e., the energy per unit area per unit time) is given by

$$\vec{S} = \begin{cases} \frac{1}{\mu_0}(\vec{E} \times \vec{B}) & \text{(SI units)} \\ \frac{c}{4\pi}(\vec{E} \times \vec{B}) & \text{(Gaussian units)} \end{cases} .$$

(c) (20 points) The wave is incident on a free electron of charge e and mass m whose average position is the origin. Calculate the total average scattered power that is radiated away by the electron. Recall the Larmor formula which relates the power P radiated by a point charge q that has an acceleration a

$$P = \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2a^2}{c^3} & \text{(SI units)} \\ \frac{2}{3}\frac{q^2a^2}{c^3} & \text{(Gaussian units)} \end{cases} .$$

(d) (10 points) The total scattering cross section σ is the ratio of the scattered power from the electron to the incident power per unit area. Compute σ and express your answer in terms of the classical electron radius r_e defined by

$$r_e = \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{e^2}{m_e c^2} & \text{(SI units)} \\ \frac{e^2}{m_e c^2} & \text{(Gaussian units)} \end{cases} .$$

Next we work out the angular dependence of the radiate power. Recall that for dipole radiation, the scattering cross section per unit spatial angle is given by

$$\frac{d\sigma}{d\Omega} = A \sin^2\theta ,$$

where θ is the angle between the direction to the observer and the acceleration vector, and $d\Omega = 2\pi \sin\theta d\theta$ is the spatial angle (integrated over the the azimuthal direction).

(e) (20 points) Determine the constant A in terms of the classical electron radius r_e by doing the appropriate integral and comparing with your result in (d).

(f) (20 points) Assume the observer is in the $x-z$ plane at an angle α to the z axis. What is angular dependence of the scattered radiation? How does this answer change if the incident radiation were polarized in the y direction?

Explicit Forms of Vector Operations

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian
$(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical
(ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \mathbf{e}_3\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical
(r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right] + \mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)\right]$$

35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

	J	J	...
	M	M	...
m_1 m_2			
m_1 m_2	Coefficients		
. .			
. .			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2×1/2

m_1	m_2	1, +1		
+1/2	+1/2	1		
m_1	m_2	1, 0	0, 0	
+1/2	−1/2	1/2	1/2	
−1/2	+1/2	1/2	−1/2	
m_1	m_2	1, −1		
−1/2	−1/2	1		

1×1/2

m_1	m_2	3/2, +3/2	
+1	+1/2	1	
m_1	m_2	3/2, +1/2	1/2, +1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3
m_1	m_2	3/2, −1/2	1/2, −1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3
m_1	m_2	3/2, −3/2	
−1	−1/2	1	

2×1

m_1	m_2	3, +3		
+2	+1	1		
m_1	m_2	3, +2	2, +2	
+2	0	1/3	2/3	
+1	+1	2/3	−1/3	
m_1	m_2	3, +1	2, +1	1, +1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10
m_1	m_2	3, 0	2, 0	1, 0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10
m_1	m_2	3, −1	2, −1	1, −1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5
m_1	m_2	3, −2	2, −2	
−1	−1	2/3	1/3	
−2	0	1/3	−2/3	
m_1	m_2	3, −3		
−2	−1	1		

1×1

m_1	m_2	2, +2		
+1	+1	1		
m_1	m_2	2, +1	1, +1	
+1	0	1/2	1/2	
0	+1	1/2	−1/2	
m_1	m_2	2, 0	1, 0	0, 0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3
m_1	m_2	2, −1	1, −1	
0	−1	1/2	1/2	
−1	0	1/2	−1/2	
m_1	m_2	2, −2		
−1	−1	1		

3/2×1

m_1	m_2	5/2, +5/2		
+3/2	+1	1		
m_1	m_2	5/2, +3/2	3/2, +3/2	
+3/2	0	2/5	3/5	
+1/2	+1	3/5	−2/5	
m_1	m_2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6
m_1	m_2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2
m_1	m_2	5/2, −3/2	3/2, −3/2	
−1/2	−1	3/5	2/5	
−3/2	0	2/5	−3/5	
m_1	m_2	5/2, −5/2		
−3/2	−1	1		

2×1/2

m_1	m_2	5/2, +5/2	
+2	+1/2	1	
m_1	m_2	5/2, +3/2	3/2, +3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5
m_1	m_2	5/2, +1/2	3/2, +1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5
m_1	m_2	5/2, −1/2	3/2, −1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5
m_1	m_2	5/2, −3/2	3/2, −3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5
m_1	m_2	5/2, −5/2	
−2	−1/2	1	

3/2×1/2

m_1	m_2	2, +2	
+3/2	+1/2	1	
m_1	m_2	2, +1	1, +1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4
m_1	m_2	2, 0	1, 0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2
m_1	m_2	2, −1	1, −1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4
m_1	m_2	2, −2	
−3/2	−1/2	1	

QUALIFYING EXAMINATION, Part 2

1:00 pm – 5:00 pm, Friday September 1, 2017

Attempt all parts of all four problems.

Please begin your answer to each problem on a separate sheet, write your 3 digit code and the problem number on each sheet, and then number and staple together the sheets for each problem. Each problem is worth 100 points; partial credit will be given.

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Problem 1: Quantum Mechanics I

In bound state problems, degeneracy of energy levels is associated with symmetry, sometimes not immediately apparent. For the Coulomb potential, with a rotational $SO(3)$ symmetry, the degeneracy with respect to the angular momentum quantum number l ($l = 0, 1, 2, \ldots, n-1$, where n is the principal quantum number), is associated with a larger "hidden" SO(4) symmetry.

This problem explores the connection between degeneracy and symmetry for the two-dimensional isotropic harmonic oscillator, whose Hamiltonian is

$$H = \frac{1}{2m}\left(p_1^2 + p_2^2\right) + \frac{1}{2}m\omega^2\left(x_1^2 + x_2^2\right) = \hbar\omega\left(\sum_{j=1,2} a_j^\dagger a_j + 1\right) ,$$

where $a_j = \sqrt{\frac{m\omega}{2\hbar}}\left(x_j + \frac{ip_j}{m\omega}\right)$ and $[a_j, a_k^\dagger] = \delta_{jk}$. It has a rotational $SO(2)$ symmetry (corresponding to rotations around an axis perpendicular to the plane of the oscillator).

(a) (10 points) Study the energy levels of this Hamiltonian by splitting the Hamiltonian according to $H = H_1 + H_2$, where each part describes a one-dimensional oscillator. What is $[H_1, H_2]$? Can one find simultaneous eigenstates of the H_j [with eigenvalues $\hbar\omega\left(n_j + \frac{1}{2}\right)$, $n_j \geq 0$ integers]?

(b) (15 points) The total energy eigenvalues of the full system can be labeled by the quantum number $n \equiv n_1 + n_2$. What are the possible values of n? What is the energy E_n associated with quantum number n? What is the degeneracy of the level associated with quantum number n?

(c) (15 points) To explain the degeneracy in (b) via symmetry, define the four operators $A_{jk} = a_j^\dagger a_k$, where $A_{11} + A_{22} = H/(\hbar\omega) - 1$. Three other *Hermitian* linear combinations can be formed from the A_{jk}. One of them is $A_{11} - A_{22}$. What are the other two? Do all three commute with H? If so, they can be taken to be the generators of the symmetry you are looking for.

(d) (10 points) One of the Hermitian combinations in (c) can be identified with the angular momentum $\hbar L \equiv x_1 p_2 - x_2 p_1$. Find which is the particular combination that gives L. The three generators in (c) do not mutually commute. Thus only one of them can be simultaneously diagonalized with H. Take it to be L.

(e) (20 points) The commutation relations among the three Hermitian generators you found in (c) are that of a three-dimensional angular momentum. That is, $[J_x, J_y] = iJ_z$, etc., where $J_z = L/2$, and J_x and J_y are proportional to the other two combinations you found in (c). (Note that only J_z is a physical angular momentum here.) Show that the operators J_x and J_y, defined with appropriate proportionality coefficients, satisfy

$[J_x, J_y] = iJ_z$.

Hint: it is convenient to first calculate the commutators $[A_{jk}, A_{lm}]$.

(f) (15 points) The simultaneous eigenvalues of $\mathbf{J}^2$ are then $j(j+1)$, where it can be shown that $j = n/2$ (you need not show this). From the properties of the angular momentum algebra, what is the degeneracy of each energy level in terms of j? Compare to your result in part (b). What are the possible eigenvalues of L in each degenerate level with quantum number n?

(g) (15 points) What is the larger symmetry group responsible for the degeneracy?

Problem 2: Quantum Mechanics II

Consider a three-level system described by the Hamiltonian

$$\frac{H}{\hbar} = \Omega_1 \left|e\right\rangle \left\langle g_1\right| + \Omega_1^* \left|g_1\right\rangle \left\langle e\right| + \Omega_2 \left|e\right\rangle \left\langle g_2\right| + \Omega_2^* \left|g_2\right\rangle \left\langle e\right| \ ,$$

where Ω_1 and Ω_2 are complex coupling parameters, and the three levels are described by the normalized states $|g_1\rangle$, $|g_2\rangle$ and $|e\rangle$.

(a) (25 points) For real $\Omega_1 = \Omega_2 = \Omega > 0$, find all the eigenenergies and eigenstates.

Hint: describe the Hamiltonian by a 3×3 matrix.

(b) (25 points) For real $\Omega_1 = \Omega_2 = \Omega > 0$, assume an initial state $|\psi(0)\rangle = |g_1\rangle$ and calculate the state $|\psi(t)\rangle$ at time t. Find the minimal time $T > 0$ when $|\psi(T)\rangle$ is given by $|g_2\rangle$.

(c) (20 points) For general complex coupling parameters Ω_1 and Ω_2, show that there is always an eigenstate with zero eigenenergy. Write an explicit expression for the normalized eigenstate of zero energy.

(d) (30 points) We may generalize the result in part (c) to an n-level system ($n \geq 3$) described by the Hamiltonian

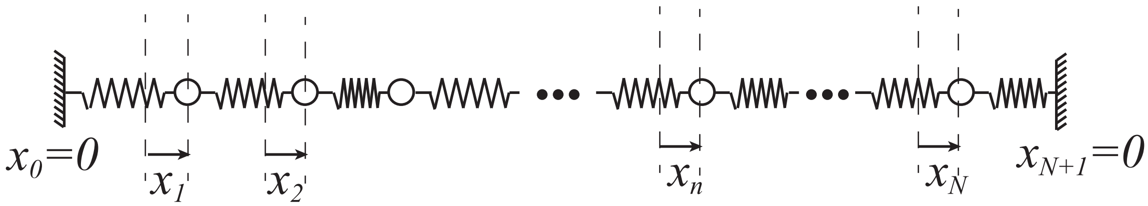
$$\frac{H}{\hbar} = \sum_{j=1}^{n-1} \left[\Omega_j \left|e\right\rangle \left\langle g_j\right| + \Omega_j^* \left|g_j\right\rangle \left\langle e\right|\right] \ ,$$

where the n levels are described by the normalized states $|g_j\rangle$ ($j = 1, \ldots, n-1$) and $|e\rangle$. Show that there are $(n-2)$ eigenstates of H with zero eigenenergy.

Hint: consider a general state $|\phi\rangle$ that is a linear combination of the n states describing the n levels, i.e., $\phi = \sum_{i=1}^{n-1} c_i|g_i\rangle + c_e|e\rangle$, and find the conditions that the coefficients c_i ($i = 1, \ldots, n-1$) and c_e must satisfy for $|\phi\rangle$ to be an eigenstate of zero energy.

Problem 3: Statistical Mechanics I

Consider a linear chain of N beads as shown in the figure. Each bead is connected to its two neighbors via springs of unstretched length a and spring constant κ, except for bead 1 and bead N, which are connected, also via springs of spring constant κ and unstretched length a to two fixed points, separated by a distance $(N+1)a$. The chain is in thermal equilibrium with a heat bath at temperature T.



The potential energy of the chain is

$$V(x_1, x_2, \ldots, x_N) = \sum_{n=0}^{N} \frac{1}{2}\kappa(x_{n+1} - x_n)^2 \ ,$$

where x_n is the displacement of bead n from its position in the completely unstretched configuration of the chain. Note that $x_0 = x_{N+1} = 0$. In the following, we denote by $\mathbf{x}$ the N-dimensional vector of displacements $x_n (n = 1, \ldots, N)$.

In solving this problem, assume *classical* statistical mechanics.

(a) (15 points) Write an expression for the *classical* probability density $P(\mathbf{x})$ (up to an overall normalization) for a general potential $V(\mathbf{x})$. Justify your answer.

(b) (20 points) We can rewrite the potential energy above as

$$V(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{V}\mathbf{x},$$

where $\mathbf{V}$ is the potential energy matrix (of constants), and $\mathbf{x}^T$ is the transpose of the column vector $\mathbf{x}$. The matrix $\mathbf{V}$ can be diagonalized by an orthogonal transformation via the transformation matrix $\mathbf{R}$ whose elements are

$$R_{mn} = \sqrt{\frac{2}{N+1}} \sin\left(\frac{\pi mn}{N+1}\right) \ .$$

Then the corresponding vector of normal coordinates $\mathbf{y}$ is given by $\mathbf{y} = \mathbf{R}\mathbf{x}$, i.e., $y_m = \sum_{n=1}^{N} R_{mn} x_n (m = 1, \ldots, N)$. The eigenvalues of $\mathbf{V}$ are given by

$$\lambda_m = 2\kappa \left(1 - \cos \frac{m\pi}{N+1}\right) , \quad (m = 1, 2, 3, ..., N) .$$

Find the normalized probability density $P(y_m)$ of a normal coordinate y_m.

(c) (20 points) Determine the mean square amplitude of each normal coordinate, i.e., determine $\langle y_m^2 \rangle$ for each m.

(d) (10 points) What is the correlation between different normal mode coordinates, i.e., what is $\langle y_m y_n \rangle$, when $m \neq n$?

(e) (35 points) Determine the mean square displacement of bead n, i.e., $\langle x_n^2 \rangle$. Sketch your solution as a function of n.

The following relation may be useful (for $1 \leq n \leq N$)

$$\sum_{m=1}^{N} \sin^2 \left(\frac{\pi m n}{N+1}\right) \frac{1}{1 - \cos\left(\frac{m\pi}{N+1}\right)} = (N+1)n \left(1 - \frac{n}{N+1}\right) .$$

Problem 4: Statistical Mechanics II

Consider a gas of N identical fermionic atoms of spin $1/2$ and mass m in a container of volume V. The walls of the container have N_0 adsorption sites, each of which can adsorb one atom. Assume that the energy of an atom adsorbed to the wall is $-\Delta$, with $\Delta > 0$, and the energy of the atoms inside the container is the ordinary non-relativistic kinetic energy. The system is at equilibrium at temperature T and chemical potential μ.

You might find the following function of the fugacity $z \equiv e^{\beta\mu}$ (where $\beta = \frac{1}{kT}$) useful

$$f_{3/2}(z) \equiv \frac{4}{\sqrt{\pi}} \int_0^\infty dx \, \frac{x^2}{z^{-1} e^{x^2} + 1} \, .$$

This function has the property

$$f_{3/2}(z) \approx z \quad \text{for} \quad z \ll 1 \, .$$

(a) (15 points) What is the average number of atoms, $\langle N_a \rangle$, that are adsorbed to the walls as a function of the fugacity $z \equiv e^{\beta\mu}$ and temperature T?

(b) (25 points) What is the average number of atoms, $\langle N_f \rangle$, that are moving freely in the volume V as a function of the fugacity z and temperature T? The answer can be given in terms of an integral that does not need to be evaluated explicitly.

(c) (25 points) Suppose that $N > N_0$. What are the average numbers of the adsorbed and free atoms at $T = 0$? Calculate the Fermi energy of this system.

(d) (25 points) Again, suppose that $N > N_0$. What are $\langle N_a \rangle$ and $\langle N_f \rangle$ as $T \to \infty$? Note that for large temperatures the fugacity is small, i.e., $z \ll 1$.

(e) (10 points) This time, assume that $N = N_0$. What is the Fermi energy of the system?

Explicit Forms of Vector Operations

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian
$(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1 \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \right) + \mathbf{e}_2 \left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \right) + \mathbf{e}_3 \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical
(ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho} \frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial}{\partial\rho} (\rho A_1) + \frac{1}{\rho} \frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1 \left(\frac{1}{\rho} \frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z} \right) + \mathbf{e}_2 \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho} \right) + \mathbf{e}_3 \frac{1}{\rho} \left(\frac{\partial}{\partial\rho} (\rho A_2) - \frac{\partial A_1}{\partial\phi} \right)$$

$$\nabla^2\psi = \frac{1}{\rho} \frac{\partial}{\partial\rho} \left(\rho \frac{\partial\psi}{\partial\rho} \right) + \frac{1}{\rho^2} \frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical
(r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r} \frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r \sin\theta} \frac{\partial\psi}{\partial\phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_1) + \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta} (\sin\theta A_2) + \frac{1}{r \sin\theta} \frac{\partial A_3}{\partial\phi}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1 \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial\theta} (\sin\theta A_3) - \frac{\partial A_2}{\partial\phi} \right]$$

$$+ \mathbf{e}_2 \left[\frac{1}{r \sin\theta} \frac{\partial A_1}{\partial\phi} - \frac{1}{r} \frac{\partial}{\partial r} (r A_3) \right] + \mathbf{e}_3 \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_2) - \frac{\partial A_1}{\partial\theta} \right]$$

$$\nabla^2\psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial\psi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial\psi}{\partial\theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial\psi}{\partial r} \right) \equiv \frac{1}{r} \frac{\partial^2}{\partial r^2} (r\psi) \right]$$

35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			
.	.			
.	.			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2×1/2

m_1	m_2	1
		+1
+1/2	+1/2	1

m_1	m_2	1	0
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	1
		−1
−1/2	−1/2	1

1×1/2

m_1	m_2	3/2
		+3/2
+1	+1/2	1

m_1	m_2	3/2	1/2
		+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	3/2	1/2
		−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	3/2
		−3/2
−1	−1/2	1

2×1

m_1	m_2	3
		+3
+2	+1	1

m_1	m_2	3	2
		+2	+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	3	2	1
		+1	+1	+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	3	2	1
		0	0	0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	3	2	1
		−1	−1	−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	3	2
		−2	−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	3
		−3
−2	−1	1

1×1

m_1	m_2	2
		+2
+1	+1	1

m_1	m_2	2	1
		+1	+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	2	1	0
		0	0	0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	2	1
		−1	−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	2
		−2
−1	−1	1

2×1/2

m_1	m_2	5/2
		+5/2
+2	+1/2	1

m_1	m_2	5/2	3/2
		+3/2	+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	5/2	3/2
		+1/2	+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	5/2	3/2
		−1/2	−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	5/2	3/2
		−3/2	−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	5/2
		−5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	2
		+2
+3/2	+1/2	1

m_1	m_2	2	1
		+1	+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	2	1
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	2	1
		−1	−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	2
		−2
−3/2	−1/2	1

3/2×1

m_1	m_2	5/2
		+5/2
+3/2	+1	1

m_1	m_2	5/2	3/2
		+3/2	+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	5/2	3/2	1/2
		+1/2	+1/2	+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	5/2	3/2	1/2
		−1/2	−1/2	−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	5/2	3/2
		−3/2	−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	5/2
		−5/2
−3/2	−1	1