QUALIFYING EXAMINATION, Part 1

Solutions

Problem 1: Classical Mechanics I

(a) The moment of inertia is double that of each leg, which in turn is that of rod of mass $M/2$ and length l about its end point. Thus

$$I = 2\int_0^l \frac{M/2}{l} x^2 dx = \frac{Ml^2}{3} \ .$$

(b) The kinetic energy is $T = \frac{1}{2} I \dot{\delta}^2$.

(c) The potential energy, with heights measured measured from the pivot is

$$\begin{aligned} V &= -\frac{M}{2} g \frac{l}{2} \cos\theta_1 - \frac{M}{2} g \frac{l}{2} \cos\theta_2 = -\frac{M}{2} g \frac{l}{2} \left[\cos\left(\frac{\theta}{2} + \delta\right) + \cos\left(\frac{\theta}{2} - \delta\right) \right] \\ &= -\frac{M}{2} g l \cos\left(\frac{\theta}{2}\right) \cos\delta \ . \end{aligned}$$

(c) The Lagrangian is

$$L = T - V = \frac{1}{2} I \dot{\delta}^2 + \frac{1}{2} M g l \cos\left(\frac{\alpha}{2}\right) \cos\delta \ .$$

(e) The equation of motion is

$$I\ddot{\delta} + \frac{1}{2} M g l \cos\left(\frac{\alpha}{2}\right) \sin\delta = 0 \ .$$

(f) For small $\sin\delta \simeq \delta$, this becomes the equation of motion for the harmonic oscillator with frequency

$$\omega = \sqrt{\frac{\frac{M}{2} g l \cos\left(\frac{\alpha}{2}\right)}{I}} = \sqrt{\frac{3g \cos\left(\frac{\alpha}{2}\right)}{2l}} \ .$$

Problem 2: Classical Mechanics II

(a) The transformation equations are given by

$$x = r\cos\theta\ ,\quad y = r\sin\theta\ ,\quad z = r\cot\alpha\ .$$

The cartesian velocities are

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}\ ,\quad \dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}\ ,\quad \dot{z} = \dot{r}\cot\alpha\ ,$$

and the kinetic energy is

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) = \frac{1}{2}m\left[\dot{r}^2(1+\cot^2\alpha) + r^2\dot{\theta}^2\right] = \frac{1}{2}m\left(\frac{\dot{r}^2}{\sin^2\alpha} + r^2\dot{\theta}^2\right)\ .$$

Alternatively, we could have started from the general expression for T in cylindrical coordinates The potential energy is given by

$$U = mgz = mgr\cot\alpha\ .$$

The Lagrangian is

$$L = T - U = \frac{1}{2}m\left(\frac{\dot{r}^2}{\sin^2\alpha} + r^2\dot{\theta}^2\right) - mgr\cot\alpha\ .$$

(b) We determine the generalized conjugate momenta from the Lagrangian

$$p_r = \frac{\partial L}{\partial \dot{r}} = \frac{m\dot{r}}{\sin^2\alpha}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}\ .$$

The Hamiltonian is given by

$$H(r,\theta,p_r,p_\theta) = p_r\dot{r} + p_\theta\dot{\theta} - L\ .$$

Using the expressions of p_r and p_θ above, we find

$$H(r,\theta,p_r,p_\theta) = \frac{p_r^2\sin^2\alpha}{2m} + \frac{p_\theta^2}{2mr^2} + mgr\cot\alpha\ .$$

(c) Hamilton's equations of motion are

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r\sin^2\alpha}{m}\ ,$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}\ ,$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = -mg\cot\alpha + \frac{p_\theta^2}{mr^3} \ ,$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \ .$$

(d) p_θ is conserved since θ is a cyclic coordinate (i.e., it does not appear explicitly in H). H is also conserved since it (or the Lagrangian) does not depend explicitly on time. $p_\theta = mr^2\dot{\theta}$ is the angular momentum of the particle. Since the transformation equations do not depend explicitly on time the potential is velocity-independent, H is equal to the total energy $H = T + U$.

(e) For $r = r_0$, $\dot{r} = 0$ and, using the first equation of motion, $p_r = 0$. It the follows from the third equation of motion that

$$\frac{p_\theta^2}{mr_0^3} = mg\cot\alpha \ .$$

Solving for $p_\theta \equiv L_0$, we find

$$L_0 = m\sqrt{gr_0^3\cot\alpha} \ .$$

Using the expression for p_θ part (b), we have $L_0 = p_\theta = mr_0^2\omega_0$ (where $\omega_0 = \dot{\theta}$). Combining with the above equation, we find

$$\omega_0 = \sqrt{\frac{g}{r_0}\cot\alpha} \ .$$

(f) Using the third equation of motion (for p_r) together with the expression for p_r in part (b), we find

$$\dot{p}_r = \frac{m\ddot{r}}{\sin^2\alpha} = -mg\cot\alpha + \frac{L_0^2}{mr^3} \ ,$$

where we have used the conservation of $p_\theta = L_0$. This equation can also be derived directly from Lagrange's equation of motion for r, using the substitution $mr^2\dot{\theta} = L_0$.

Substituting $r = r_0 + \delta r$ in the equation above, we find

$$\frac{m\ddot{\delta r}}{\sin^2\alpha} = -mg\cot\alpha + \frac{L_0^2}{m(r_0+\delta r)^3}$$

or

$$\ddot{\delta r} \approx -g\sin^2\alpha\cot\alpha + \frac{L_0^2\sin^2\alpha}{m^2r_0^3}\left(1 - \frac{3\delta r}{r_0}\right) = -\left(\frac{3g\sin\alpha\cos\alpha}{r_0}\right)\delta r \ ,$$

where we used the Taylor expansion given in the hint, and substituted the value of L_0 found in part (e). Thus the angular frequency of small oscillations is

$$\omega = \sqrt{\frac{3g\sin\alpha\cos\alpha}{r_0}} \ .$$

Note that the ratio $\omega/\omega_0 = \sqrt{3}\sin\alpha$ is independent of r_0.