QUALIFYING EXAMINATION, Part 2

Solutions

Problem 1: Electromagnetism I

(a) The total charge of the sphere is 0 by symmetry. Therefore at $r \gg a$, the system resembles a dipole, and the leading order radial dependence of the potential is $\sim 1/r^2$. The complete behavior (including the angular dependence) is given by

$$V(r,\theta) \sim \frac{1}{r^2}\cos\theta \,.$$

(b) Since the potential must vanish as $r \to \infty$, we have $A_l = 0$. Also, since the leading order behavior at $r \gg a$ is $\sim 1/r^2$, we have $B_0 = 0$.

$$V(r,\theta) = \sum_{l=1} \frac{B_l}{r^{l+1}} P_l(\cos\theta)\,.$$

V satisfies the following boundary conditions at $r = a$

$$V(r=a,\theta) = \begin{cases} V & 0 \le \theta \le \pi/2 \\ -V & \pi/2 \le \theta \le \pi \end{cases} \,.$$

Using the expansion for V taking the scalar product with $P_m(\cos\theta)$ on both sides, we find

$$\sum_l \int d(\cos\theta)\,\frac{B_l}{a^{l+1}} P_l(\cos\theta)P_m(\cos\theta) = V\left[\int_0^{\pi/2} d(\cos\theta)P_m(\cos\theta) - \int_{\pi/2}^{\pi} d(\cos\theta)P_m(\cos\theta)\right] \,.$$

Using the orthogonality of the Legendre polynomials and $P_l(-x) = (-)^l P_l(x)$, we find

$$\frac{2}{(2m+1)a^{m+1}} B_m = [1 + (-1)^{m+1}]V \int_0^1 dx\, P_m(x)\,.$$

It follows that $B_m = 0$ for m even. Thus the two lowest order terms are $m = 1$ and $m = 3$. Solving the above equations for $m = 1$ and $m = 3$ using the integrals in the hints, we find

$$V(r,\theta) = V\left[\frac{3}{2}\left(\frac{a}{r}\right)^2 P_1(\cos\theta) - \frac{7}{8}\left(\frac{a}{r}\right)^4 P_3(\cos\theta) + \ldots\right] \,.$$

(c) For $r < a$, all of the $B_{l,\text{in}}$ must vanish to ensure that the potential is finite at $r = 0$. Matching the potentials inside the shell with the potential outside the shell at $r = a$ yields

$$A_{l,\text{in}} = B_{l,\text{out}} a^{-2l-1} \,.$$

It follows that $A_{l,\mathrm{in}} = 0$ for all odd values of l.

(d) Using the values of $B_{l,\mathrm{out}}$ found in (c) and the matching condition in (c), we find

$$V(r,\theta) = V\left[\frac{3}{2}\left(\frac{r}{a}\right)P_1(\cos\theta) - \frac{7}{8}\left(\frac{r}{a}\right)^3 P_3(\cos\theta) + \ldots\right] .$$

(e) The net surface charge density equals the discontinuity in the normal component of the electric field across the shell

$$\sigma = \epsilon_0(\mathbf{E}_{\mathrm{out},\perp} - \mathbf{E}_{\mathrm{in},\perp}) .$$

The radial component of the electric field outside is

$$E_{r,\mathrm{out}}(r=a) = \frac{V}{a}\left[3P_1(\cos\theta) - \frac{7}{2}P_3(\cos\theta) + \ldots\right] ,$$

while the radial component inside is

$$E_{r,\mathrm{in}}(r=a) = -\frac{V}{a}\left[\frac{3}{2}P_1(\cos\theta) - \frac{21}{8}P_3(\cos\theta) + \ldots\right] .$$

Putting these together, we have

$$\sigma = \frac{\epsilon_0 V}{a}\left[\frac{9}{2}P_1(\cos\theta) - \frac{49}{8}P_3(\cos\theta) + \ldots\right] .$$

Problem 2: Electromagnetism II

In the following we use SI units but also give the answers in cgs units.

(a) In the quasi-static limit, the electric field is the instantaneous field generated by the charge distribution on the plates. This field is constant between the plates along the z-axis. By integrating a Gaussian pillbox centered on the lower plate, we find

$$\mathbf{E} = \frac{1}{\epsilon_0}\sigma\hat{\mathbf{z}} = \frac{1}{\epsilon_0}\sigma_0\cos(\omega t)\hat{\mathbf{z}} \ .$$

In cgs units

$$\mathbf{E} = 4\pi\sigma_0\cos(\omega t)\hat{\mathbf{z}} \ .$$

(b) The displacement current density is

$$\mathbf{J}_d = \epsilon_0\frac{\partial \mathbf{E}}{\partial t} = \dot{\sigma}\hat{\mathbf{z}} = -\omega\sigma_0\sin(\omega t)\hat{\mathbf{z}} \ .$$

By symmetry, $\mathbf{B}$ will point in the azimuthal direction $\hat{\phi}$. Applying Ampere's law to a circular loop of radius r concentric with the z axis, we find

$$2\pi r B_\phi = \mu_0 \pi r^2 J_d \ ,$$

or

$$\mathbf{B} = -\frac{\mu_0\omega r}{2}\sigma_0\sin(\omega t)\hat{\phi} \ .$$

Note that $B_\phi/E_z \sim \omega r/c^2 \sim \mathcal{O}(\epsilon)$ for $r < a$.

In cgs, we have

$$\mathbf{B} = -\frac{4\pi\omega r}{2c}\sigma_0\sin(\omega t)\hat{\phi} \ .$$

(c) The Poynting vector is

$$\mathbf{S} = (1/\mu_0)(\mathbf{E}\times\mathbf{B}) = \frac{1}{2\epsilon_0}\omega r\sigma_0^2\sin(\omega t)\cos(\omega t)\hat{\mathbf{r}} \ .$$

The energy flux through the cylindrical surface is then given by

$$(2\pi r d)S_{r=d} = (\pi r^2 d)\frac{1}{\epsilon_0}\omega\sigma_0^2\sin(\omega t)\cos(\omega t) \ .$$

In cgs, we obtain

$$\mathbf{S} = 2\pi\omega r\sigma_0^2\sin(\omega t)\cos(\omega t)\hat{\mathbf{r}} \ .$$

The energy flux is

$$(2\pi r d)S_{r=d} = (\pi r^2 d)4\pi\omega\sigma_0^2\sin(\omega t)\cos(\omega t) \ .$$

(d) The energy density is given by

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) \ .$$

We ignore the energy density of the induced magnetic field since its contribution is higher order in ϵ. We find for the rate of change of energy stored inside the cylinder

$$(\pi r^2 d) du/dt = (\pi r^2 d)\frac{1}{2}\epsilon_0 dE^2/dt = (\pi r^2 d)\frac{1}{\epsilon_0}\sigma\dot{\sigma} = -(\pi r^2 d)\frac{1}{\epsilon_0}\omega\sigma_0^2 \sin(\omega t)\cos(\omega t) \ .$$

In cgs units, the energy density is

$$u = \frac{1}{8\pi}\left(E^2 + B^2\right) \ .$$

This yields

$$(\pi r^2 d) du/dt = (\pi r^2 d)\frac{1}{8\pi} dE^2/dt = (\pi r^2 d) 4\pi\sigma\dot{\sigma} = -(\pi r^2 d) 4\pi\omega\sigma_0^2 \sin(\omega t)\cos(\omega t) \ .$$

Comparing with the result in (c), we see that the rate of change of energy is minus the energy flux through the surface (as is expected from energy conservation).

(d) The time-dependent induced magnetic field generates, by Faraday's law, an $\mathcal{O}(\epsilon^2)$ correction to the electric field of part (a). By symmetry we expect the induced electric field $\mathbf{E}_{\rm ind}$ to point along the z-axis and to depend only on r, i.e., $\mathbf{E}_{\rm ind} = E_{\rm ind}(r)\hat{\mathbf{z}}$.

We take a rectangular loop of width dr between the two plates with two sides at distances r and $r + dr$ from the center parallel to the z axis. Using Faraday's law we have

$$[E_{\rm ind}(r) - E_{\rm ind}(r + dr)]d = -\frac{d}{dt}[B_\phi(r)(dr)d] \ ,$$

or

$$\frac{dE_{\rm ind}}{dr} = \frac{dB_\phi}{dt} \ .$$

Using the result for B_ϕ in (b) and integrating the equation from $r = 0$ to r, we find

$$E_{\rm ind}(r) - E_{\rm ind}(r = 0) = -\frac{1}{4}\mu_0(\omega r)^2\sigma_0 \cos(\omega t)$$

The ratio between the induced electric field and the original electric field is $\frac{1}{4}(\omega r/c)^2$, i.e., of order ϵ^2.

In cgs units

$$E_{\rm ind}(r) - E_{\rm ind}(r = 0) = -\pi(\omega r/c)^2\sigma_0 \cos(\omega t) \ .$$