Problem 1: Quantum Mechanics I

(a) The probability $P_{ab}(t)$ is given by

$$P_{ab}(t) = \int_a^b dx |\psi(x,t)|^2 ,$$

so that

$$dP_{ab}/dt = \int_a^b dx \left( \psi^* \partial_t \psi + c.c. \right) .$$

Using the Schrödinger equation for $\psi(x,t)$

$$i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} \partial_x^2 \psi(x,t) + V(x)\psi(x,t) ,$$

and its complex conjugate, we find

$$dP_{ab}/dt = i\frac{\hbar}{2m} \int_a^b dx \left( \psi^* \partial_x^2 \psi - \psi \partial_x^2 \psi^* \right) .$$

Integrating both terms by parts

$$dP_{ab}/dt = i\frac{\hbar}{2m} \left( \psi^* \partial_x \psi |_a^b - \psi \partial_x \psi^* |_a^b + \int_a^b dx \left[ -\partial_x \psi^* \partial_x \psi + \partial_x \psi \partial_x \psi^* \right] \right) .$$

The integrand on the r.h.s. vanishes, and we find

$$dP_{ab}/dt = J(a,t) - J(b,t) ,$$

where

$$J(x,t) = \frac{i\hbar}{2m} \left( \psi \partial_x \psi^* - \psi^* \partial_x \psi \right)$$

is the probability current.

(b) We determine $A$ from the normalization of the wave function

$$\int_{-\infty}^{\infty} dx |\psi(x,t)|^2 = \int_{-\infty}^{\infty} dx A^2 \exp(-2c|x|) = 2A^2 \int_0^{\infty} dx \exp(-2cx) = A^2/c = 1 .$$

Therefore

$$A = \sqrt{c} .$$
\[ \langle \hat{x} \rangle (t) = \int dx x |\psi(x,t)|^2 = 0 \text{ since } |\psi(x,t)|^2 \text{ is an even function. For } \langle \hat{x}^2 \rangle (t) \text{ we find } \]
\[ \langle \hat{x}^2 \rangle (t) = \int_{-\infty}^{\infty} dx x^2 |\psi(x,t)|^2 = 2c \int_{0}^{\infty} dx x^2 \exp(-2cx) = \frac{1}{2c^2}. \]

(c) The time dependence of \( \psi(x,t) \) in part (b) is of the form \( e^{-iEt/\hbar} \). Substituting this form in the Schrödinger equation, we find that the spatial part of \( \psi(x,t) \) obeys the time-independent Schrödinger equation. Therefore \( \psi \) is a stationary state and all expectation values are time independent
\[ \langle \hat{O} \rangle (t) = \langle \hat{O} \rangle (t = 0) . \]

(d) Substituting the form of the wave function in (b) into the Schrödinger equation, we find for \( x > 0 \) or \( x < 0 \)
\[ i\hbar \partial_t \psi = E\psi = -\hbar^2 c^2/(2m)\psi + V(x)\psi = -\hbar^2/(2m)\partial_x^2 \psi + V(x)\psi , \]
or
\[ \left( E + \frac{\hbar^2 c^2}{2m} \right) \psi = V(x)\psi . \]
Thus \( V(x) \) must be constant for \( x \neq 0 \). Since \( V(x) \to 0 \) at large \( |x| \), we conclude that \( V(x) = 0 \) for \( x \neq 0 \) and also that its energy \( E = -\hbar^2 c^2/(2m) < 0 \), i.e., \( \psi \) is a bound state of \( V(x) \).

Note that \( V(x) \) cannot be zero at \( x = 0 \) because then it would be zero everywhere, and a free particle does not have bound states.

Next, we integrate the Schrödinger equation around \( x = 0 \) over a narrow interval \(-\epsilon < x < \epsilon \) for \( \epsilon > 0 \) and then take the limit \( \epsilon \to 0 \). We find
\[ E \int_{-\epsilon}^{\epsilon} dx \psi = -\frac{\hbar^2}{2m} \int_{-\epsilon}^{\epsilon} dx \partial_x^2 \psi + \int_{-\epsilon}^{\epsilon} dx \psi V(x) = -\frac{\hbar^2}{2m} \partial_x \psi |_{-\epsilon}^{\epsilon} + \psi(0,t) \int_{-\epsilon}^{\epsilon} dx V(x) . \]
Since the wave function \( \psi \) is continuous, The expression on the l.h.s. vanishes in the limit \( \epsilon \to 0 \). Using the actual form of \( \psi(x,t) \) from part (b) in the r.h.s. of the above equation, we find
\[ 0 = A \frac{\hbar^2 c}{m} + A \int_{-\epsilon}^{\epsilon} dx V(x) , \]
or
\[ \int_{-\epsilon}^{\epsilon} dx V(x) = -\frac{\hbar^2 c}{m} . \]
Since \( V(x) = 0 \) for \( x \neq 0 \) and its integral around \( x = 0 \) is a constant, it must be proportional to the Dirac \( \delta \) function. Therefore
\[ V(x) = -\frac{\hbar^2 c}{m} \delta(x) . \]
Problem 2: Quantum Mechanics II

(a) There are four states, one 2s state, $|2s, m_l = 0\rangle$, and three 2p states, $|2p, m_l = +1\rangle$, $|2p, m_l = 0\rangle$, and $|2p, m_l = -1\rangle$.

(b) The potential has odd parity, and therefore the matrix elements between 2s states or 2p states must vanish. As the zero component of a spherical tensor of rank one, $V$ can only connect states with the same $m_l$. Thus the only non-vanishing matrix elements are between $|2s, m_l = 0\rangle$ and $|2p, m_l = 0\rangle$. The matrix element $\Delta E_{\pm} = \pm 3\epsilon a_0$. In first-order degenerate perturbation theory, we diagonalize the corresponding $2 \times 2$ matrix of $V$:

$$
\begin{pmatrix}
0 & 3\epsilon a_0 \\
3\epsilon a_0 & 0
\end{pmatrix}.
$$

The energy shifts are given by the eigenvalues of this matrix:

$$\Delta E_{\pm} = \pm 3\epsilon a_0 .$$

The energies of the $|2p, m_l = +1\rangle$ and $|2p, m_l = -1\rangle$ states are not shifted.

(c) These are the eigenvectors of the matrix in (b)

$$|\pm\rangle = \frac{1}{\sqrt{2}} (|2s, m_l = 0\rangle \pm |2p, m_l = 0\rangle) .$$

(d) In the limit $\epsilon a_0 \ll \delta$, we can ignore the splitting $\delta$ between the $2s_{1/2}$ states and the $2p_{1/2}$ states, and solve the problem in first-order degenerate perturbation theory. Since $V$ conserves $m_j$, the only non-vanishing matrix elements of $V$ are between $|2s_{1/2}, m_j = +1/2\rangle$ and $|2p_{1/2}, m_j = +1/2\rangle$, and also between $|2s_{1/2}, m_j = -1/2\rangle$ and $|2p_{1/2}, m_j = -1/2\rangle$. The Stark shift in each case is linear in $\epsilon$, as determined by degenerate perturbation theory.

(e) In the limit $\epsilon a_0 \ll \delta$, we can consider the Stark shift separately for the $2s_{1/2}$ states and the $2p_{1/2}$ states. For each such set of degenerate states (with $m_j = \pm 1/2$) there are no off-diagonal matrix elements, and the diagonal matrix elements also vanish (since $V$ has odd parity). Therefore there is no contribution in first-order perturbation theory, and one must proceed to second-order perturbation theory. The Stark shift of each level is thus quadratic in $\epsilon$. Non-degenerate perturbation theory can be used.