QUALIFYING EXAMINATION, Part 1

Solutions

Problem 1: Classical Mechanics I

(a) Setting all lengths of the problem with respect to the center of pulley A and letting the distance between the center of pulley A and pulley B be $l_a - x_1$, and the distance between the center of pulley B and m_3 equal $l_b - x_2$, we have

$$x_{m1} = x_1$$

$$x_{m2} = (l_a - x_1) + x_2$$

$$x_{m3} = (l_a - x_1) + (l_b - x_2) \; .$$

The velocities of the three masses are

$$v_{m1} = \dot{x_1}$$

$$v_{m2} = -\dot{x_1} + \dot{x_2}$$

$$v_{m3} = -\dot{x_1} - \dot{x_2} \; .$$

The kinetic energy T of the system is then

$$T = \frac{1}{2} m_1 \dot{x_1}^2 + \frac{1}{2} m_2 (-\dot{x_1} + \dot{x_2})^2 + \frac{1}{2} m_3 (-\dot{x_1} - \dot{x_2})^2 \; .$$

(b) The heights of the three masses, as measured from the center of pulley A are

$$h_1 = -x_1$$

$$h_2 = -[(l_a - x_1) + x_2] = -l_a + x_1 - x_2$$

$$h_3 = -[(l_a - x_1) + (l_b - x_2)] = -l_a + x_1 - l_b + x_2 \; .$$

The total gravitational potential energy is then given by

$$V = m_1 g(-x_1) + m_2 g(-l_a + x_1 - x_2) + m_3 g(-l_a + x_1 - l_b + x_2) \; .$$

Omitting the constant terms, which are not relevant to changes in potential energy, we have

$$V = -g x_1 (m_1 - m_2 - m_3) - g x_2 (m_2 - m_3) \; ,$$

which will be used for the remainder of the problem.

(c) The Lagrangian is

$$L = T - V = \frac{1}{2}m_1\dot{x_1}^2 + \frac{1}{2}m_2(-\dot{x_1}+\dot{x_2})^2 + \frac{1}{2}m_3(-\dot{x_1}-\dot{x_2})^2 + gx_1(m_1-m_2-m_3) + gx_2(m_2-m_3) \; .$$

(d) The equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x_1}}\right) - \frac{\partial L}{\partial x_1} = 0 \; ,$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x_2}}\right) - \frac{\partial L}{\partial x_2} = 0 \; .$$

We have

$$\frac{\partial L}{\partial \dot{x_1}} = \dot{x_1}(m_1 + m_2 + m_3) + \dot{x_2}(m_3 - m_2) \; ; \quad \frac{\partial L}{\partial x_1} = g(m_1 - m_2 - m_3) \; .$$

The first equation of motion is therefore

$$(m_1 + m_2 + m_3)\ddot{x_1} + (m_3 - m_2)\ddot{x_2} - g(m_1 - m_2 - m_3) = 0 \; .$$

Similarly

$$\frac{\partial L}{\partial \dot{x_2}} = \dot{x_1}(m_3 - m_2) + \dot{x_2}(m_3 + m_2) \; ; \quad \frac{\partial L}{\partial x_2} = g(m_2 - m_3) \; .$$

The second equation of motion is therefore

$$(m_3 - m_2)\ddot{x_1} + (m_3 + m_2)\ddot{x_2} - g(m_2 - m_3) = 0 \; .$$

(e) To modify our Lagrangian to take into account the mass of pulley A, we need to add the kinetic energy of pulley A

$$T_{\text{pulley}} = \frac{1}{2}I\omega^2 \; ,$$

where ω is the angular velocity of the pulley, and I is its moment of inertia. I is given by

$$I = \frac{1}{2}Mr_A^2 \; .$$

Since the rope does not slip on the pulley, we have $\omega = \frac{\dot{x_1}}{r_A}$ and

$$T_{\text{pulley}} = \frac{1}{4}M\dot{x_1}^2 \; .$$

Since the height of pulley A does not change, there is no modification to the gravitational potential energy. The new Lagrangian is therefore given by

$$L = T - V = \frac{1}{4}M\dot{x_1}^2 + \frac{1}{2}m_1\dot{x_1}^2 + \frac{1}{2}m_2(-\dot{x_1}+\dot{x_2})^2 + \frac{1}{2}m_3(-\dot{x_1}-\dot{x_2})^2 + gx_1(m_1-m_2-m_3) + gx_2(m_2-m_3) \; .$$

Problem 2: Classical Mechanics II

(a) For equal spring constant, the bar hangs horizontally at equilibrium. The equilibrium position is then given by setting $F_{springs} = F_{gravity}$ at equal extension z_{eq}:

$$Mg = 2kz_{eq} \Rightarrow z_{eq} = \frac{Mg}{2k} \ .$$

(b) For the two normal modes specified in the problem, we have the following relations: for mode (1) where the bar remains horizontal and the center-of-mass oscillates in z, we have $z_1 = z_2$; and for mode (2) where the center of mass remains fixed and the bar rotates, we have $z_1 = -z_2$.

For (1), the equation of motion for $z = z_1 = z_2$ is

$$M\ddot{z} = -2kz$$
$$\Rightarrow \ddot{z} = -\frac{2k}{M}z \ .$$

For (2), we can find the angular acceleration from the torque around the center of mass of the bar. First, the moment of inertia for rotation of the bar about its center is

$$I = M\int_{-d/2}^{d/2} x^2 dx = \frac{1}{12}Md^2 \ .$$

For the rotation angle relative to the horizontal, $z = z_1 = -z_2 = \left(\frac{d}{2}\right)\theta$ for small angles θ. The torque from each spring is $N = -(d/2)kz$, giving a total torque of $N_{tot} = -dkz = -\frac{d^2}{2}k\theta$. The equation of motion for θ is then

$$\frac{1}{12}Md^2\ddot{\theta} = -\frac{d^2}{2}k\theta \quad \Rightarrow \quad \ddot{\theta} = -\frac{6k}{M}\theta \ .$$

The equations of motion describe simple harmonic motion with frequencies of

$$\omega_1 = \sqrt{\frac{2k}{M}}, \qquad \omega_2 = \sqrt{\frac{6k}{M}} \ .$$

(c) For small oscillations about the equilibrium, the linear terms must vanish and the potential energy is

$$V = \frac{k}{2}\left(z_1^2 + z_2^2\right) \ .$$

This could also be worked out explicitly from the full expression for the potential energy

$$V = \frac{k}{2}\left((z_1 + z_{eq})^2 + (z_2 + z_{eq})^2\right) - \frac{Mg}{2}(z_1 + z_2)$$
$$= \frac{k}{2}\left[(z_1 + \frac{Mg}{2k})^2 + (z_2 + \frac{Mg}{2k})^2 - \frac{Mg}{k}(z_1 + z_2)\right] = \frac{k}{2}\left(z_1^2 + z_2^2\right) + V_0 \ ,$$

where we used that the position of the center of mass is $z_{cm} = -\frac{1}{2}(z_1 + z_2)$ and we can drop the constant offset V_0 related to the potential at the equilibrium.

The total kinetic energy is $T = T_{cm} + T_{rot} = \frac{1}{2}M\dot{z}_{cm}^2 + \frac{1}{2}I\dot{\theta}^2$. We can use the expression for z_{cm} and I above. Measuring θ relative to the horizontal, then for small angles, $(z_1 - z_2) = \theta d$, so:

$$T = \frac{1}{8}M\left(\dot{z}_1 + \dot{z}_2\right)^2 + \frac{1}{24}Md^2\left(\frac{\dot{z}_1 - \dot{z}_2}{d}\right)^2 = \frac{1}{6}M\left(\dot{z}_1{}^2 + \dot{z}_2{}^2 + \dot{z}_1\dot{z}_2\right) .$$

(d) The frequencies of the normal modes are determined from

$$\det(\mathbf{V} - \omega^2\mathbf{M}) = 0 .$$

From part (c) we have

$$\mathbf{V} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} , \quad \mathbf{M} = \frac{M}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} ,$$

where the matrices are defined by $V = \frac{1}{2}\sum V_{ij}\eta_i\eta_j$ and $T = \frac{1}{2}\sum M_{ij}\dot{\eta}_i\dot{\eta}_j$, where $\eta_i = \{z_1, z_2\}$

We then have

$$\begin{vmatrix} k - \frac{M\omega^2}{3} & -\frac{M\omega^2}{6} \\ -\frac{M\omega^2}{6} & k - \frac{M\omega^2}{3} \end{vmatrix} = 0$$

$$\Rightarrow \left(k - \frac{M\omega^2}{3}\right)^2 - \left(\frac{M\omega^2}{6}\right)^2 = 0$$

$$\Rightarrow \omega^4 - 8\frac{k}{M}\omega^2 + 12\frac{k^2}{M^2} = 0 .$$

which gives:

$$\omega^2 = \frac{k}{M}\left(\frac{8 \pm \sqrt{64 - 48}}{2}\right) = \frac{k}{M}(4 \pm 2) .$$

This reproduces the values for ω_1 and ω_2 found in part (c).

(e) For normal mode 1

$$\begin{pmatrix} k - \frac{M\omega_1^2}{3} & -\frac{M\omega_1^2}{6} \\ -\frac{M\omega_1^2}{6} & k - \frac{M\omega_1^2}{3} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0 \Rightarrow \frac{k}{3M}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0 ,$$

which implies $z_1 = z_2$ in this mode, i.e., it is the mode for oscillations of the center of mass as expected.

For normal mode 2

$$\begin{pmatrix} k - \frac{M\omega_2^2}{3} & -\frac{M\omega_2^2}{6} \\ -\frac{M\omega_2^2}{6} & k - \frac{M\omega_2^2}{3} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0 \Rightarrow \frac{-k}{M}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0 ,$$

which implies $z_1 = -z_2$ in this mode, i.e., it is the mode for rotational oscillations with fixed center of mass, also as expected.