QUALIFYING EXAMINATION, Part 2

Solutions

Problem 1: Electromagnetism I

(a) Since our system has spherical symmetry the electric field and potential have, respectively, the form $\mathbf{E} = E(r)\,\hat{r}$ and $\varphi = \varphi(r)$.

We use the integral form of Gauss's law to determine the electric field. For $r < R$, we have

$$E(r)(4\pi r^2) = \frac{Q_{enc}}{\varepsilon_0} = -\frac{q}{\varepsilon_0}\frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3} = -\frac{q}{\varepsilon_0}\left(\frac{r}{R}\right)^3 \ ,$$

or

$$E(r) = -\left(\frac{q}{4\pi\varepsilon_0 R^3}\right) r \ .$$

For $r > R$, we find

$$E(r) = -\frac{q}{4\pi\varepsilon_0}\frac{1}{r^2} \ .$$

Since $\mathbf{E} = -\nabla\varphi$, we can find the electrostatic potential by integrating the electric field. Up to an integration constant, the potential is given by $\varphi(r) = -\int E_r(r)dr$. For $r > R$, using the boundary condition $\phi(r) \to 0$ for $r \to \infty$, we find

$$\varphi(r) = -\frac{q}{4\pi\varepsilon_0}\frac{1}{r} \ .$$

For $r < R$, we find

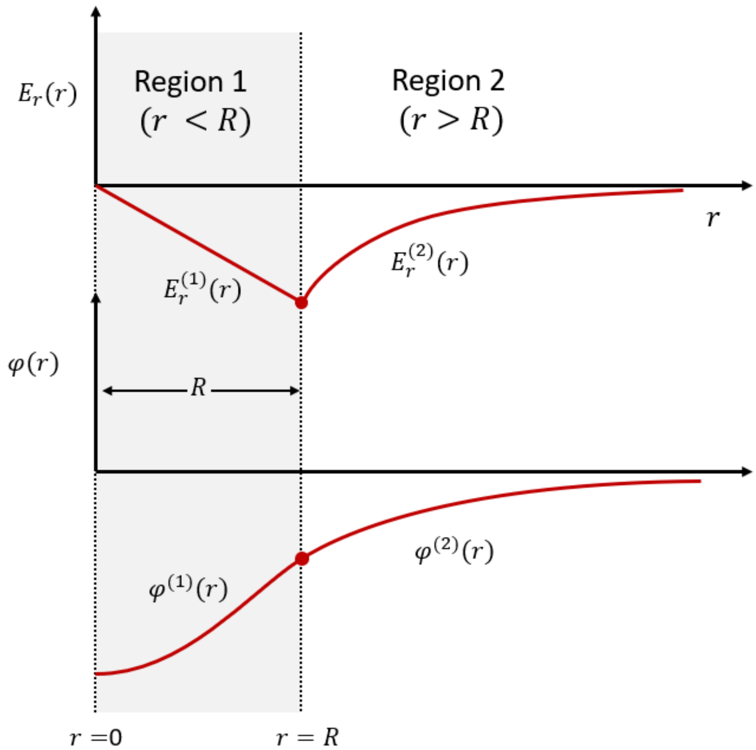
$$\varphi(r) = +\frac{1}{2}\left(\frac{q}{4\pi\varepsilon_0 R^3}\right) r^2 + C \ ,$$

where C is an integration constant. We determine C using the continuity condition for the electrostatic potential at $r = R$, and find $C = -\frac{3}{2}\, q\,(4\pi\varepsilon_0 R)^{-1}$. Therefore, the potential for $r < R$ is

$$\varphi(r) = -\frac{1}{2}\frac{q}{4\pi\varepsilon_0 R}\left[3 - \left(\frac{r}{R}\right)^2\right] \ .$$

Alternatively, this problem can be solved by finding the solution to Poisson's (Laplace's) equation for $r < R$ ($r > R$). In this method, we would find the solution to the differential equation $\nabla^2\varphi(r) = -\rho(r)/\varepsilon_0$. The electric field would then computed using $\mathbf{E} = -\nabla\varphi$.

(b) The electric field and the potential are sketched in the figure.



(c) The stable equilibrium position is at the origin $r = 0$, at which the force due to the field of the charged sphere vanishes.

(d) In equilibrium the total force vanishes

$$\mathbf{F}_{sphere} + \mathbf{F}_{ext} = 0 \ .$$

or

$$qE(r)\hat{r} + qE_0\hat{x} = 0 \ .$$

Using the expression for $E(r)$ from part (a) inside the sphere, we find for the new equilibrium position

$$\mathbf{d} = E_0 \frac{4\pi\varepsilon_0 R^3}{q} \hat{x} \ .$$

We can also minimize the total potential energy obtained by adding $-E_0 x$ to the electrostatic potential inside the sphere.

(e) Outside the sphere, we can take its charge $-q$ to be at the origin. Hence, the superposition this negative charge with the positive point charge $(+q)$ gives the system a dipole moment of

$$\mathbf{p} = q\,\mathbf{d} = (4\pi\,\varepsilon_0 R^3) E_0 \hat{x} \ .$$

(f) Comparing the expression in (e) with with $\mathbf{p} = \alpha\mathbf{E}$, we find

$$\alpha = 4\pi\,\varepsilon_0 R^3\ .$$

Interestingly, the polarizability depends only on the radius of our classical atom; the larger the radius the larger the polarizability, meaning that our classical atom exhibits a larger response to a given applied field.

Problem 2: Electromagnetism II

(a) Using Biot-Savart Law (in SI units):

$$\mathrm{d}\vec{B} = \frac{\mu_0 I \, \mathrm{d}\vec{l} \times \hat{r}}{4\pi r^2} \, ,$$

where $\mathrm{d}\vec{l}$ is the differential directed length of the current, and r is the distance from the differential current to the observation point. Using the cylindrical symmetry and integrating along the loop (equivalent to multiplying by by $2\pi R$), we find

$$B_z = \frac{\mu_0 I R^2}{2\left(R^2 + z^2\right)^{3/2}} \qquad [\mathrm{SI}] \, .$$

or

$$B_z = \frac{2\pi I R^2}{c\left(R^2 + z^2\right)^{3/2}} \qquad [\mathrm{Gaussian}] \, .$$

(b)

$$B_z = \frac{\mu_0 I R^2}{2} \left\{ \frac{1}{\left[R^2 + (H - z)^2\right]^{3/2}} + \frac{1}{\left[R^2 + (H + z)^2\right]^{3/2}} \right\} \qquad [\mathrm{SI}] \, .$$

(c)

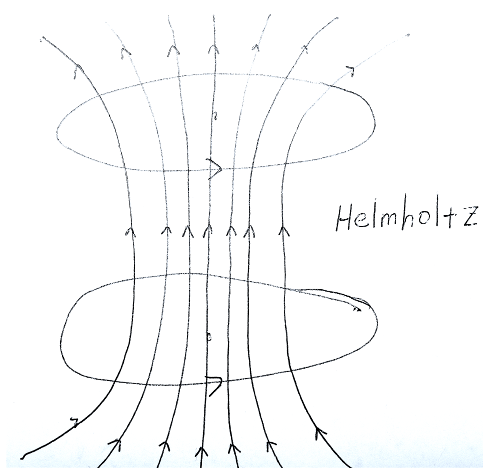


Figure 1: Helmholtz

(d)

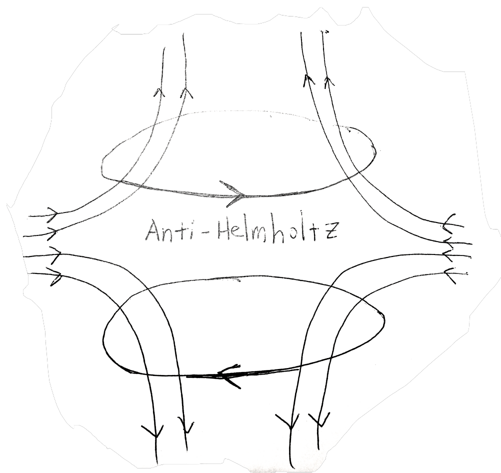


Figure 2: Anti-Helmholtz

(e) Near $z = 0$, we have $\partial B_z/\partial z = \eta I_0$.

Using Maxwell's equation

$$\nabla \cdot \mathbf{B} = \partial B_z/\partial z + \partial B_y/\partial y + \partial B_x/\partial x = 0 \ .$$

By cylindrical symmetry $\partial B_y/\partial y = \partial B_x/\partial x$, so $\partial B_x/\partial x = \partial B_y/\partial y = -\eta I_0/2$.

(f) By Faraday's Law, the electromotive force along a loop is $\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$,where Φ is the magnetic flux through the loop. Here, the loop is a ring of wire of radius r where the field is nearly constant over the area of the ring. From part (e) $B_z(t) = Iz = \eta I_0 z \cos(\omega_0 t)$.

The flux through the small loop is $\Phi(z,t) = \pi r^2 \eta I_0 z \cos(\omega_0 t)$. Thus the induced emf is $\mathcal{E} = \pi r^2 \eta I_0 \omega_0 z \sin(\omega_0 t)$, and the current is

$$I_{\text{ring}} = \frac{\mathcal{E}}{Z_0} = \frac{\pi r^2 \eta I_0 \omega_0 z \sin(\omega_0 t)}{Z_0} \ .$$