QUALIFYING EXAMINATION, Part 4

Solutions

Problem 1: Statistical Mechanics I

(a) The one-body distribution for a classical ideal gas with a single-particle Hamiltonian $h = \frac{\mathbf{p}^2}{2m} + V_{\text{trap}}(\mathbf{r})$ is $f(\mathbf{r}, \mathbf{p}) = \mathcal{N} \exp[-\beta \frac{\mathbf{p}^2}{2m} - \beta V_{\text{trap}}(\mathbf{r})]$, where $\mathcal{N}$ is a normalization factor. The density distribution is then $n(r) = \int \mathrm{d}^2\mathbf{p}\ f(\mathbf{r}, \mathbf{p}) = C e^{-\beta V_{\text{trap}}(r)}$. The prefactor C is such that $2\pi C \int_0^R r\mathrm{d}r\ e^{-\frac{\beta}{2} m\omega^2 r^2} = N$. Using the formula in the reminder, we find

$$C = \frac{\beta m \omega^2 N}{2\pi} \frac{1}{1 - e^{-\frac{\beta}{2} m\omega^2 R^2}} .$$

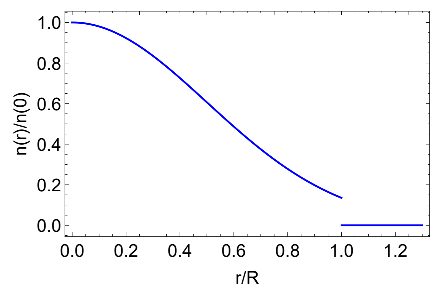

Figure 1: $n(r)/n(0)$ versus r/R.

(b)

$$\begin{aligned} h_{\text{rot}} &= \frac{\mathbf{p}^2}{2m} + V_{\text{trap}}(r) - \mathbf{\Omega}\cdot(\mathbf{r} \times \mathbf{p}) = \frac{\mathbf{p}^2}{2m} + V_{\text{trap}}(r) - \mathbf{p}\cdot(\mathbf{\Omega} \times \mathbf{r}) \\ &= \frac{1}{2m} \left(\mathbf{p} - m(\mathbf{\Omega} \times \mathbf{r})\right)^2 - \frac{m}{2}(\mathbf{\Omega} \times \mathbf{r})^2 + V_{\text{trap}}(r) = \frac{1}{2m} \left(\mathbf{p} - m(\mathbf{\Omega} \times \mathbf{r})\right)^2 - V_{\text{eff}}(\mathbf{r}) \end{aligned}$$

Since the particles are confined in a plane orthogonal to $\mathbf{\Omega}$, we have $\mathbf{\Omega} \times \mathbf{r} = \Omega r$. We thus find that $V_{\text{eff}} = \frac{1}{2} m(\omega^2 - \Omega^2) r^2$ for $r < R$ (and ∞ otherwise). We see that the centrifugal force results in a deconfining radial harmonic potential with frequency Ω.

(c) Following part (a) and replacing h_{lab} by h_{rot}, one finds after integrating the one-body distribution over momenta (at any fixed $\mathbf{r}$, the term $m\mathbf{\Omega} \times \mathbf{r}$ just shifts $\mathbf{p}$ by a constant) that the equilibrium density is $n_\Omega(r) = D e^{-\beta V_{\text{eff}}(r)}$. The calculation of D is almost identical to the one for C

$$D = \frac{N}{\pi R^2} \frac{\alpha}{1 - e^{-\alpha}} .$$

The resulting density distribution for the three rotation regimes are shown in Fig. 2.

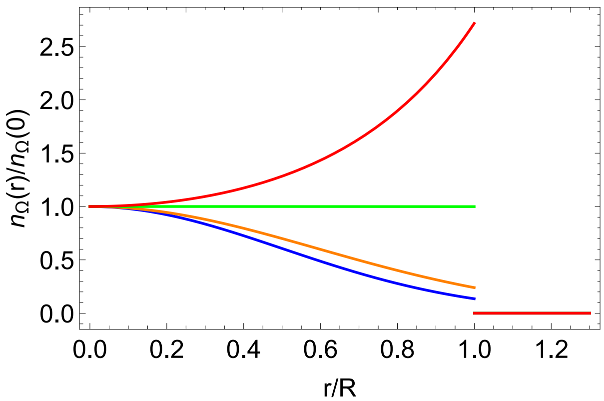

Figure 2: $n_\Omega(r)/n_\Omega(0)$ for no ($\Omega = 0$, blue, same as Fig.1), slow ($\Omega \ll \omega$, orange), fast ($\Omega = \omega$, green) and ultrafast ($\Omega \gg \omega$, red) rotations.

(d) Following the expression in the reminders, the partition function of the rotating gas is

$$\mathcal{Z} = \frac{1}{h^{2N}N!}\left(\int \mathrm{d}^2\mathbf{r}\mathrm{d}^2\mathbf{p}\exp\left[-\frac{\beta}{2m}\left(\mathbf{p} - m(\mathbf{\Omega}\times\mathbf{r})\right)^2 - \beta V_{\text{eff}}(\mathbf{r})\right]\right)^N .$$

Carrying out first the integration over momenta, the $\mathbf{\Omega}\times\mathbf{r}$ term is just a constant shift, and we find

$$\mathcal{Z} = A(\beta, N)\left(\int \mathrm{d}^2\mathbf{r}\exp\left[-\beta V_{\text{eff}}(\mathbf{r}_i)\right]\right)^N .$$

where $A(\beta, N)$ is independent of R. The integration over spatial coordinates is very similar to the calculation of the normalization constants above:

$$\int \mathrm{d}^2\mathbf{r}\exp\left[-\beta V_{\text{eff}}(\mathbf{r}_i)\right] = 2\pi R^2\int_0^1 ue^{-\alpha u^2}\mathrm{d}u = 2\pi R^2\frac{1-e^{-\alpha}}{2\alpha} .$$

Since R^2/α is a constant independent of R, we find

$$\mathcal{Z} = \mathcal{Z}_0\left(1 - e^{-\alpha}\right)^N ,$$

where $\mathcal{Z}_0$ is independent of R.

(e) Using the chain rule $\frac{\partial}{\partial R} = \frac{2\alpha}{R}\frac{\partial}{\partial \alpha}$, the force exerted on the disk wall is

$$F = \frac{2\alpha N}{\beta R}\frac{\partial}{\partial \alpha}[\log(1 - e^{-\alpha})] = \frac{2N}{\beta R}\frac{\alpha}{e^{\alpha} - 1} ,$$

and the pressure is

$$P = \frac{N}{\beta \pi R^2}\frac{\alpha}{e^{\alpha} - 1} .$$

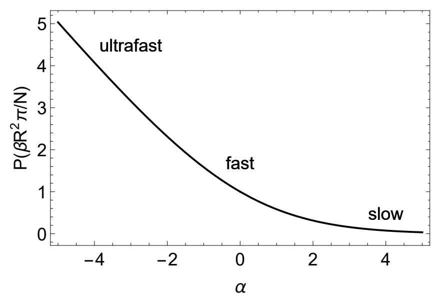

Figure 3: The (dimensionless) pressure per particle $P\frac{\beta\pi R^2}{N}$ versus the rotation parameter α at fixed temperature.

P is sketched in Fig. 3, and its interpretation is straightforward when one notices that the pressure can be simply expressed as a function of the particle density at the walls $P = n_\Omega(R)k_\mathrm{B}T$. This is the form of the ideal gas equation of state with the local density at the walls.

For fast rotation ($\alpha = 0$), we recover the equation of state of an ideal gas with a uniform density $N/(\pi R^2)$. For ultrafast rotation ($\alpha \to -\infty$), $P \approx \frac{N}{2\pi} m\Omega^2$. In this limit, particles are squashed against the walls, and the pressure is the total centrifugal force $m\Omega^2 R \times N$ divided by the total length of the (1D) walls $2\pi R$, independent of the temperature.

Problem 2: Statistical Mechanics II

(a) The spectrum has a symmetry under $\epsilon \to -\epsilon$, i.e. $D(-\epsilon) = D(\epsilon)$. The density of states for positive values of ϵ is obtained from the matching

$$\begin{aligned} D(\epsilon)d\epsilon &= D(\vec{k})d^2\vec{k} \\ &= g\left(\frac{L_x}{2\pi}\right)\left(\frac{L_y}{2\pi}\right)(2\pi k)dk \\ &= \frac{A}{\pi}\frac{\epsilon}{\hbar v}\frac{d\epsilon}{\hbar v}. \end{aligned}$$

Taking into account the symmetry, we obtain the full result

$$D(\epsilon) = \frac{A}{\pi\hbar^2 v^2}|\epsilon|,$$

whose plot looks like the $|\epsilon|$ function.

(b) The Fermi-Dirac distribution states that the probability to find a fermion in a single spin state with energy ϵ is given by

$$n(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} + 1}.$$

If we assume that $\mu = 0$, then the probability to find a hole in a single spin state at energy $-\epsilon$ is

$$n_h(-\epsilon) = 1 - n(-\epsilon) = 1 - \frac{1}{e^{\beta(-\epsilon)} + 1} = \frac{e^{\beta(-\epsilon)} + 1 - 1}{e^{\beta(-\epsilon)} + 1} = \frac{1}{e^{\beta\epsilon} + 1} = n(\epsilon).$$

(c) Assuming $\mu = 0$, the number of holes in the valence band is

$$N_h = \int_{-\infty}^{0} n_h(\epsilon) D(\epsilon) d\epsilon = \int_0^{\infty} n_h(-\epsilon) D(-\epsilon) d\epsilon = \int_0^{\infty} n(\epsilon) D(\epsilon) d\epsilon = N_p.$$

(d) The total internal energy of excitations above the $T = 0$ state is given by

$$\begin{aligned} U(T) - U(0) &= \textstyle\int_{-\infty}^{0} (-\epsilon) n_h(\epsilon) D(\epsilon) d\epsilon + \int_0^{\infty} \epsilon n(\epsilon) D(\epsilon) d\epsilon \\ &= 2 \textstyle\int_0^{\infty} \epsilon n(\epsilon) D(\epsilon) d\epsilon = \frac{2A}{\pi\hbar^2 v^2} \int_0^{\infty} \frac{\epsilon^2}{e^{\beta\epsilon}+1} d\epsilon \\ &= \textstyle\frac{2A}{\pi\hbar^2 v^2} \int_0^{\infty} \frac{(x/\beta)^2}{e^x+1} \frac{dx}{\beta} = \frac{2A(k_B T)^3}{\pi\hbar^2 v^2} \int_0^{\infty} \frac{x^2}{e^x+1} dx \end{aligned}$$

Using the integral given in the problem, we find

$$U(T) - U(0) = \frac{2A(k_B T)^3}{\pi\hbar^2 v^2} 2(1 - 2^{-2})\zeta(3) = \frac{3\zeta(3) A (k_B T)^3}{\pi\hbar^2 v^2}.$$

(e) The heat capacity is

$$\begin{aligned} C_A(T) &= \left.\frac{\partial U(T)}{\partial T}\right|_A \\ &= \frac{9\zeta(3) A k_B^3}{\pi\hbar^2 v^2} T^2, \end{aligned}$$

so the exponent is $\alpha = 2$.