QUALIFYING EXAMINATION, Part 1

1:00 PM – 4:00 PM, Thursday September 1, 2011

Attempt all parts of all four problems.

Please begin your answer to each problem on a separate sheet, write your 3 digit code and the problem number on each sheet, and then number and staple together the sheets for each problem. Each problem is worth 100 points; partial credit will be given.

Calculators may NOT be used.

Problem 1: Mathematical Methods

Consider the following differential equation for the function $y = y(x)$

$$x\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 0 \ .$$

(a) How many linearly independent solutions do you expect? (15 points)

(b) Look for solutions of the form

$$y(x) = \sum_{n=0}^{\infty} x^{n+s} c_n \ ,$$

where s is a real number and c_n are constants. Find the allowed values for s. By summing the series for the largest value of s, show that one solution is given by $y_1 = c_0 x e^x$. (40 points)

Note: other values of s do not lead to a new solution (you do not have to show that).

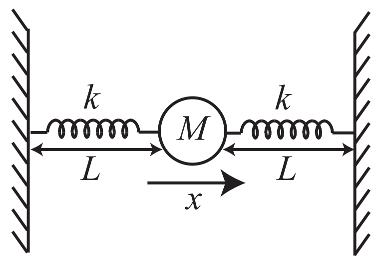
(c) Find another solution of the form $y_2(x) = y_1(x)F(x)$. Express F in terms of an indefinite integral that cannot be evaluated in closed form. However, show that for small x

$$y_2(x) = ce^x\left(1 + x\ln x - \frac{x^2}{2} + \frac{x^3}{12} + \mathcal{O}(x^4)\right) \ ,$$

where c is a constant. (45 points)

Problem 2: Classical Mechanics

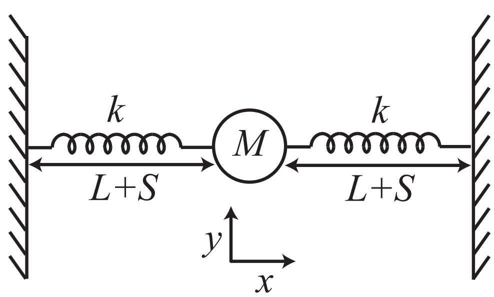
A mass M is connected on each side by a massless spring, each of spring constant k (see figure), and it executes small oscillations along the x-axis. At equilibrium the springs are unstretched at length L.



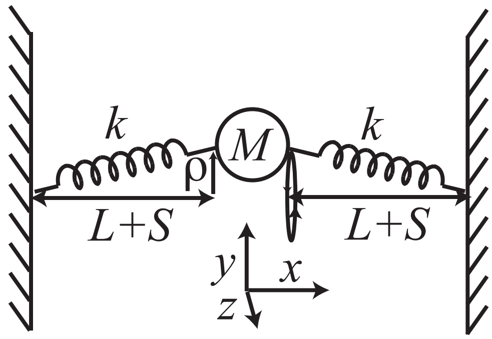
(a) For motion along the x-axis, determine the Lagrangian $\mathcal{L}$ of the system. Compute the Euler-Lagrange equation for x and find the angular frequency ω for the oscillation. (30 points)

(b) Assume the springs are stretched by an amount S at equilibrium (see second figure) and allow the mass M to execute small oscillations along both the x and y axes (the y-axis is perpendicular to the springs). Find the potential energy $V(x, y) - V(0, 0)$ to second order in x and y, and compute the normal frequencies and normal modes. (40 points)

Hint: use $x, y \ll L + S \equiv \lambda$ and the expansion $\sqrt{1+\xi} = 1 + \frac{1}{2}\xi - \frac{1}{8}\xi^2 + \ldots$ (for $\xi \ll 1$). Feel free to use general arguments regarding linear terms in the potential.

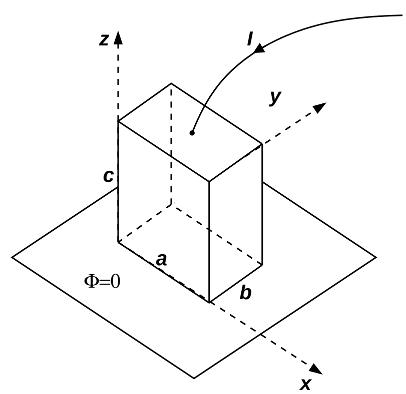


(c) Assume the mass in (b) undergoes circular motion with a small radius ρ in the $y - z$ plane perpendicular to the x axis (i.e., the $x = 0$ plane) (see third figure). Use Newton's law to determine the rotation frequency ω around the $x-$axis. Use your results in (b) to argue that such circular motion is indeed a solution to the equations of motion in the limit of small oscillations and to determine the frequency ω. (30 points)


k
M
k
ρ
L+S
L+S
y
x
z

Problem 3: Electromagnetism I

A block of a semiconductor (see figure) with conductivity σ is attached with its bottom face ($z = 0$) to a metal plate, which is held at zero potential ($\Phi_{plate} = 0$). Assume that the component of the electric field perpendicular to each of the vertical sides of the block ($x = 0, x = a, y = 0, y = b$) vanishes. A boundary condition similar to that of the sides is imposed at the top face, except for a point at its center where a thin wire carrying a current I is attached. The charge density inside the block is $\rho = 0$, and assume $\epsilon = \mu = 1$.



(a) Write down the general solution for the electric potential Φ inside the block. (25 points)

(b) Employ the boundary conditions for the vertical sides ($x = 0, x = a, y = 0, y = b$) to constrain parameters of the general solution. (25 points)

(c) Use the boundary condition for the bottom plate (at $z = 0$) to constrain further the parameters. Write down the solution after steps (b) and (c). (25 points)

(d) Without doing the explicit calculation, explain how you could use the boundary condition for the top face ($z = c$) to determine the final solution. (25 points)

Hint: use Ohm's law $E_z = I/\sigma$ at the point contact and model the contact of the wire by δ functions in x and y. Thus $\partial\Phi/\partial z|_{z=c} = -\frac{I}{\sigma}\delta\left(x - \frac{a}{2}\right)\delta\left(y - \frac{b}{2}\right)$.

Problem 4: Electromagnetism II

An amount of charge Q is distributed uniformly over the surface of a spherical insulator of radius a. The sphere is then made to rotate about an axis (the z-axis) with constant angular frequency ω.

(a) Give an expression for the surface charge density σ and the surface current density $\vec{K}$ on the surface of the sphere. (15 points)

(b) Calculate the electric field both inside and outside the sphere. (15 points)

(c) Calculate the magnetic moment $\vec{\mu}$ of the system. (20 points)

Hint: the magnetic moment of a circular ring carrying a current is proportional to the current times its area.

(d) The magnetic field produced by this configuration is uniform along the z-axis inside the sphere ($r < a$) and dipolar outside the sphere ($r > a$). Find the exact expressions of $\vec{B}$ for $r < a$ and $r > a$ in terms of the parameters Q, a, ω. (40 points)

Hints: express $\vec{B}$ in terms magnetic scalar potentials Φ_M, $\vec{B} = -\vec{\nabla}\Phi_M$ in the regions $r < a$ and $r > a$. Use the fact that the magnetic field is uniform along the z-axis inside the sphere and dipolar outside the sphere (i.e., $\Phi_M \propto \cos\theta/r^2$ for $r > a$), and apply the boundary conditions for $\vec{B}$ across $r = a$ to determine the proportionality constants in the above expressions for Φ_M.

The gradient in spherical coordinates is given by

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \ .$$

(e) Now suppose that the angular velocity is changing as a function of time at a rate of $\dot{\omega}$. Use Faraday's law to find the azimuthal electric field E_ϕ induced on the surface of the sphere at angle θ by the time-dependent magnetic flux. (10 points)

Explicit Forms of Vector Operations

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian
$(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical
(ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \mathbf{e}_3\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical
(r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right]$$

$$+\mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)\right]$$

35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			
.	.			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2×1/2

		1
		+1
+1/2	+1/2	1

		1	0
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		1
		−1
−1/2	−1/2	1

1×1/2

		3/2
		+3/2
+1	+1/2	1

		3/2	1/2
		+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

		3/2	1/2
		−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

		3/2
		−3/2
−1	−1/2	1

2×1

		3
		+3
+2	+1	1

		3	2
		+2	+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

		3	2	1
		+1	+1	+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

		3	2	1
		0	0	0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

		3	2	1
		−1	−1	−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

		3	2
		−2	−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

		3
		−3
−2	−1	1

1×1

		2
		+2
+1	+1	1

		2	1
		+1	+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

		2	1	0
		0	0	0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

		2	1
		−1	−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

		2
		−2
−1	−1	1

3/2×1

		5/2
		+5/2
+3/2	+1	1

		5/2	3/2
		+3/2	+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

		5/2	3/2	1/2
		+1/2	+1/2	+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

		5/2	3/2	1/2
		−1/2	−1/2	−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

		5/2	3/2
		−3/2	−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

		5/2
		−5/2
−3/2	−1	1

2×1/2

		5/2
		+5/2
+2	+1/2	1

		5/2	3/2
		+3/2	+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

		5/2	3/2
		+1/2	+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

		5/2	3/2
		−1/2	−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

		5/2	3/2
		−3/2	−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

		5/2
		−5/2
−2	−1/2	1

3/2×1/2

		2
		+2
+3/2	+1/2	1

		2	1
		+1	+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

		2	1
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		2	1
		−1	−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

		2
		−2
−3/2	−1/2	1

QUALIFYING EXAMINATION, Part 2

9:00 AM – noon, Friday September 2, 2011

Attempt all parts of all four problems.

Please begin your answer to each problem on a separate sheet, write your 3 digit code and the problem number on each sheet, and then number and staple together the sheets for each problem. Each problem is worth 100 points; partial credit will be given.

Calculators may NOT be used.

Problem 1: Quantum Mechanics I

This problem is sometimes referred to as a bound-state Aharanov-Bohm effect.

An electron of charge e is confined to a ring of radius R in the $x-y$ plane centered around the z-axis. A constant magnetic field $\vec{B}$, pointing in the $\hat{z}$ direction, is confined to an infinitely long cylinder whose axis is the z-axis and whose radius is $R_0 < R$. Thus $\vec{B} = B\hat{z}$ for $\rho \leq R_0$, and $\vec{B} = 0$ for $\rho > R_0$, where $\rho^2 = x^2 + y^2$.

(a) Using Stokes' theorem along the ring of radius R, find the simplest form for the vector potential $\vec{A}$ on the ring. Take $\vec{A}$ to point along the ring and be constant in magnitude (i.e., cylindrically symmetric). (20 points)

(b) Write down the quantum-mechanical Hamiltonian for an electron with canonical momentum $\vec{p}$ (neglecting its spin) in the presence an electromagnetic vector potential $\vec{A}$. Use the principle of minimal coupling substitution $\vec{p} \rightarrow \vec{p} - \frac{e}{c}\vec{A}$ (where $\vec{p}$ is the canonical momentum). (15 points)

(c) Making use of the Hamiltonian in (b), determine the allowed bound-state energies of the electron confined to the ring. What are the corresponding eigenfunctions in terms of the angular displacement φ along the ring ? (35 points)

(d) Now suppose that there is also a weak, uniform electric field acting on the electron, pointing in the $\hat{x}$ direction: $\vec{E} = E\hat{x}$. What is the form of the interaction potential V in terms of the angular displacement φ ? What is the energy shift of each level to lowest order in perturbation theory? Assume that $(e/c)(R_0^2 B/2?)$ is *not* an integral multiple of 1/2. (30 points)

Problem 2: Quantum Mechanics II

• Two identical non-relativistic particles of mass m are confined to one dimension (the x axis). Each particle moves in a harmonic trapping potential $V(x) = \frac{1}{2}kx^2$ where $k > 0$. The coordinate operator of a harmonic oscillator can be written in terms of raising and lower operators as $\hat{x} = \sqrt{\hbar/(2m\omega)}(\hat{a} + \hat{a}^\dagger)$ where $\omega = \sqrt{k/m}$.

(a) If the two particles are spinless non-interacting bosons, find the ground-state energy E_0 and first excited state energy E_1 of the system. (15 points)

(b) Solve part (a) for spinless non-interacting fermions. (15 points)

(c) Assume the two particles are non-interacting spin-1/2 electrons. What are E_0 and E_1 and their degeneracies? (25 points)

• For the remainder of this problem, consider an additional attractive interaction which is added to the system Hamiltonian

$$V_{int}(x_1, x_2) = -\alpha k x_1 x_2 \ ,$$

where x_1 and x_2 are the particle coordinates and α is a dimensionless parameter $0 < \alpha < 1$ (parts (a)-(c) had $\alpha = 0$).

(d) For spinless bosons, find the correction to E_0 to lowest non-vanishing order in α. (25 points)

(e) Find exact, closed-form expressions (i.e., no approximations or series) for $E_0(\alpha)$ and $E_1(\alpha)$ for spinless bosons.

Hint: transform x_1, x_2 to normal coordinates. (20 points)

Problem 3: Statistical Mechanics I

In 1907, Einstein, in the first application of quantum theory to a problem other than radiation, modeled a solid body containing N atoms as a collection of $3N$ harmonic oscillators (each such atom can vibrate in three directions). This is known as Einstein's model of a solid. For simplicity, he assumed that all these oscillators have the same frequency ω and their coupling can be ignored.

(a) Calculate in closed form the canonical partition function Z for a single oscillator at temperature T. (25 points)

(b) Use your result in (a) to calculate the average energy of a single oscillator at temperature T and then determine the internal energy U of the entire solid. (25 points)

(c) Find the heat capacity $C_V \equiv \frac{\partial U}{\partial T}$ of this solid. (20 points)

(d) Sketch C_V as a function of T. Determine the behavior of C_V in the limits $kT \ll \hbar\omega$ and $kT \gg \hbar\omega$ and explain their physical meaning. Argue that the high temperature limit is valid even when the N oscillators do not have the same frequency. (30 points)

Problem 4: Statistical Mechanics II

Consider a two-dimensional gas of N non-relativistic fermions with mass m and spin s moving in a square of area A.

(a) Find the density of single-particle states $g(\epsilon)$ at energy ϵ. (20 points)

(b) Evaluate the Fermi energy ϵ_F of the gas as a function of the density of particles $\rho = N/A$. (20 points)

(c) Calculate the total energy of the gas per particle E/A at temperature $T = 0$ as a function of its density. (20 points)

(d) Using your result in (c), determine the pressure of the gas P at $T = 0$ as a function of its density. (15 points)

(e) A container is separated into two compartments by a sliding piston. Two two-dimensional Fermi gases with spin 1/2 and 3/2 but the same mass are placed in the left and right compartments, respectively. Find the ratio between the densities of the two gases at equilibrium at $T = 0$. (25 points)

Hint: use your result in (d).

Explicit Forms of Vector Operations

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian $(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical (ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \mathbf{e}_3\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical (r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right] + \mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

[Note that $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)$]

35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2			
m_1	m_2	Coefficients		
.	.			
.	.			

1/2×1/2

m_1	m_2		
	J	1	
	M	+1	
+1/2	+1/2	1	
	J	1	0
	M	0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2
	J	1	
	M	−1	
−1/2	−1/2	1	

1×1/2

m_1	m_2		
	J	3/2	
	M	+3/2	
+1	+1/2	1	
	J	3/2	1/2
	M	+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3
	J	3/2	1/2
	M	−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3
	J	3/2	
	M	−3/2	
−1	−1/2	1	

2×1

m_1	m_2			
	J	3		
	M	+3		
+2	+1	1		
	J	3	2	
	M	+2	+2	
+2	0	1/3	2/3	
+1	+1	2/3	−1/3	
	J	3	2	1
	M	+1	+1	+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10
	J	3	2	1
	M	0	0	0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10
	J	3	2	1
	M	−1	−1	−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5
	J	3	2	
	M	−2	−2	
−1	−1	2/3	1/3	
−2	0	1/3	−2/3	
	J	3		
	M	−3		
−2	−1	1		

1×1

m_1	m_2			
	J	2		
	M	+2		
+1	+1	1		
	J	2	1	
	M	+1	+1	
+1	0	1/2	1/2	
0	+1	1/2	−1/2	
	J	2	1	0
	M	0	0	0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3
	J	2	1	
	M	−1	−1	
0	−1	1/2	1/2	
−1	0	1/2	−1/2	
	J	2		
	M	−2		
−1	−1	1		

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

3/2×1

m_1	m_2			
	J	5/2		
	M	+5/2		
+3/2	+1	1		
	J	5/2	3/2	
	M	+3/2	+3/2	
+3/2	0	2/5	3/5	
+1/2	+1	3/5	−2/5	
	J	5/2	3/2	1/2
	M	+1/2	+1/2	+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6
	J	5/2	3/2	1/2
	M	−1/2	−1/2	−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2
	J	5/2	3/2	
	M	−3/2	−3/2	
−1/2	−1	3/5	2/5	
−3/2	0	2/5	−3/5	
	J	5/2		
	M	−5/2		
−3/2	−1	1		

2×1/2

m_1	m_2		
	J	5/2	
	M	+5/2	
+2	+1/2	1	
	J	5/2	3/2
	M	+3/2	+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5
	J	5/2	3/2
	M	+1/2	+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5
	J	5/2	3/2
	M	−1/2	−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5
	J	5/2	3/2
	M	−3/2	−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5
	J	5/2	
	M	−5/2	
−2	−1/2	1	

3/2×1/2

m_1	m_2		
	J	2	
	M	+2	
+3/2	+1/2	1	
	J	2	1
	M	+1	+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4
	J	2	1
	M	0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2
	J	2	1
	M	−1	−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4
	J	2	
	M	−2	
−3/2	−1/2	1	

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$