QUALIFYING EXAMINATION, Part 1

1:00 pm – 5:00 pm, Thursday September 1, 2016

Attempt all parts of all four problems.

Please begin your answer to each problem on a separate sheet, write your 3 digit code and the problem number on each sheet, and then number and staple together the sheets for each problem. Each problem is worth 100 points; partial credit will be given.

Calculators and cell phones may NOT be used.

Problem 1: Mathematical Methods

(a) (30 points) The operator implementing an infinitesimal Lorentz transformation in 1+1 dimension is

$$U(\varepsilon) = 1 - \varepsilon\tau \tag{1}$$

where ε is the infinitesimal parameter and τ is the matrix

$$\tau = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{2}$$

What is $U(\theta)$, which implements a Lorentz transformation with a finite parameter θ? (You should give U as a 2×2 matrix and write explicitly its four θ-dependent elements. You should also provide intermediate steps starting from what is given and not rely on prior knowledge of the answer.)

(b) (10 points) Argue that the integers (from $-\infty$ to ∞) form a group under addition. Name the identity element. Do they form a group under multiplication? Explain your answer.

(c) (30 points) Evaluate the following integral by contour integration

$$\int_0^\infty \frac{1 - \cos 2x}{1 + x^2} dx \ .$$

(d) (30 points) Given a basis of functions $u_n = x^n$ ($n = 0, 1, ..$) and a weight function $\mu(x) = xe^{-x}$, find the two lowest-order normalized orthogonal polynomials in the interval $0 \leq x \leq \infty$. You can use the Gram-Schmidt procedure or any other method.

Hints: A weight function defines the scalar product of two real functions $f(x)$ and $g(x)$ by

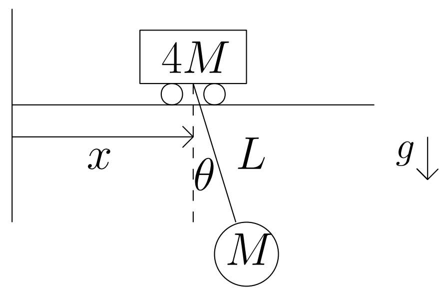
$$\langle f|g\rangle = \int_0^\infty f(x)g(x)\mu(x)dx$$

A useful integral

$$\int_0^\infty x^n e^{-x} dx = n!$$

Problem 2: Classical Mechanics

Consider a simple pendulum of mass M and length L suspended from a cart of mass $4M$. The cart can move freely along a horizontal track in the x-direction and the pendulum feels a constant downward gravitational acceleration g (see figure).



(a) (25 points) Write down the Lagrangian of this system using the position x of the cart and the angle θ between the pendulum and the vertical axis as generalized coordinates.

(b) (20 points) Write the Lagrangian you found in (a) in the limit of small oscillations around $\theta = 0$, and derive the equations of motion for x and θ in this limit.

(c) (20 points) Find the equation of motion for $x(t)$ in (b) for a solution with $\theta(t) = 0$. Write down its general solution and describe the corresponding motion.

(d) (35 points) Find the normal frequencies and normal modes of this system. Write down the most general solution to $x(t)$ and $\theta(t)$ in the limit of small oscillations. How many undetermined constants appear in this general solution?

Problem 3: Electromagnetism I

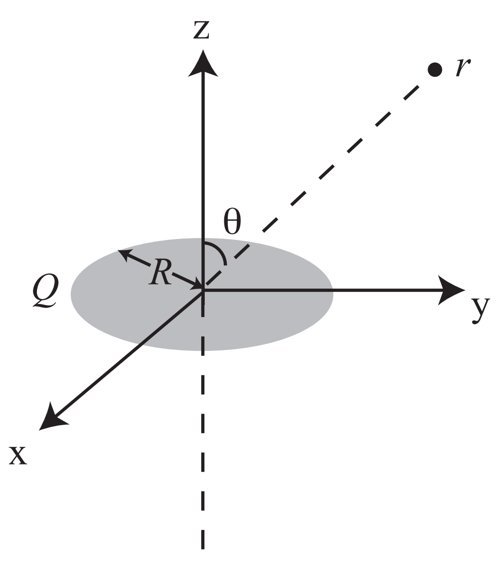
(a) (5 points) What is the electric field of a point charge q as a function of the position $\vec{r}$ where the charge is at the origin $\vec{r} = \vec{0}$?

(b) (10 points) What is the electric field of an infinite thin sheet of charge whose surface charge density [charge/area] is a constant σ.

(c) (25 points) What is the electric potential and electric field on the symmetry axis of a thin disk of charge Q and radius R as a function of the displacement along the coordinate z?

(d) (15 points) For the disk of part (c), find the electric field in the limits when $z \gg R$ and $z \ll R$ and compare with your results of parts (a) and (b).

(e) (30 points) Determine the electric potential $V(r, \theta)$ for the disk of charge Q and radius R for any $r > R$ and $0 \leq \theta \leq \pi$ (see figure).



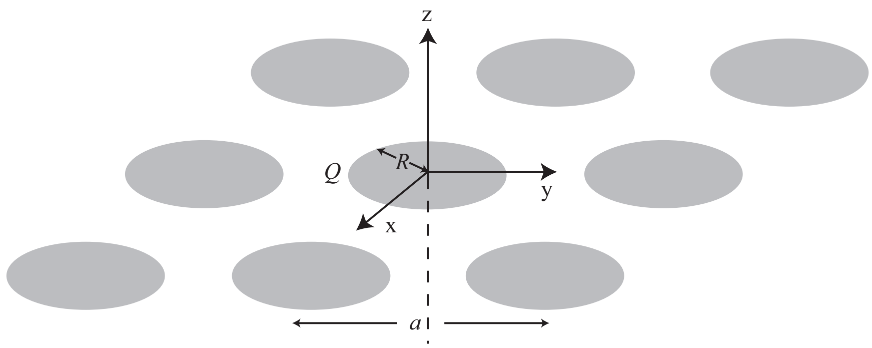
Hints: (i) The general solution of Laplace's equation with azimuthal symmetry is

$$V(r, \theta) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)} \right] P_l(\cos\theta) \ ,$$

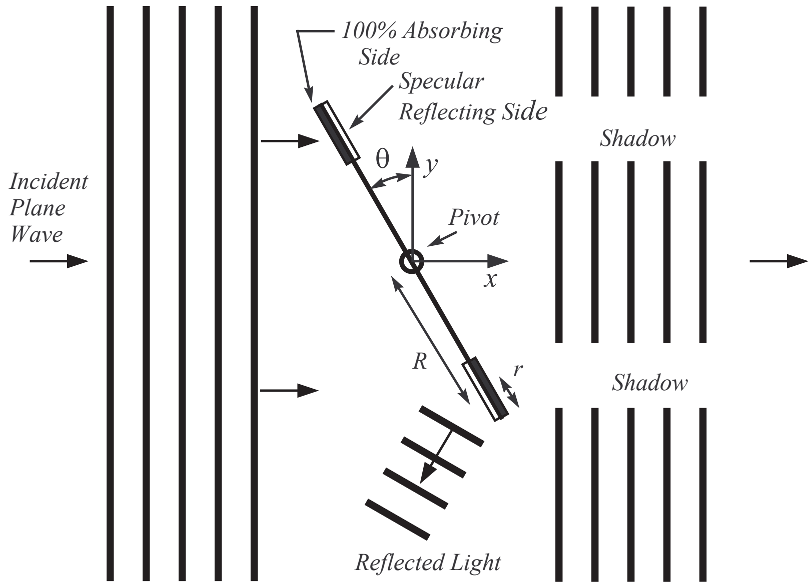
where P_l are Legendre polynomials satisfying $P_l(1) = 1$.
(ii) Use your results from part (c).
(iii) The binomial expansion for $|x| < 1$ is

$$(1 + x)^{\alpha} = \sum_{k=0}^{\infty} \begin{pmatrix} \alpha \\ k \end{pmatrix} x^k = 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!} x^2 + \frac{\alpha(\alpha - 1)(\alpha - 2)}{3!} x^3 + \ldots$$

(f) (15 points) For an infinite plane of disks, each of charge Q and radius R, whose centers are separated by $a > 2R$ (see figure), what is the magnitude and direction of the electric field at a distance $z >> a, R$ above the plane? Note that the answer to this part can be found without any detailed calculation.



Problem 4: Electromagnetism II



A massless rod can rotate without friction about the pivot point at its center (see figure). Light, electromagnetic field propagating as a plane wave, propagates from left to right, along the x axis (crests are shown as lines, but are omitted for clarity in the region of the rod). Its electric field is given by

$$\vec{E}(x,t) = E_0 \hat{y} \cos(kx - \omega t) \ ,$$

where $\vec{k} = k\hat{x}$ and E_0 is a real number. The angle between the rod and $\hat{y}$ is denoted by θ.

Centered at the ends of the rod are disks, each with one side perfectly mirror (specularly) reflecting, and the other side 100% absorbing. The disks are oriented so that light in the upper part of the rod (above the pivot) always strikes an absorptive surface, while in the lower part, it strikes a reflective surface. Each of the disks have mass m and radius r. Assume that the distance R from the pivot to the center of each disk satisfies $R \gg r$. The unit vector $\hat{z}$ is defined by $\hat{z} = \hat{x} \times \hat{y}$ (a right-handed coordinate system).

The Poynting vector, which describes the energy flux density (i.e., the energy per unit area per unit time) is given by

$$\vec{S} = \begin{cases} \frac{1}{\mu_0}(\vec{E} \times \vec{B}) & \text{(SI units)} \\ \frac{c}{4\pi}(\vec{E} \times \vec{B}) & \text{(Gaussian units)} \end{cases} ,$$

where c is the speed of light. The momentum density $\vec{p}$ carried by the wave is

$$\vec{p} = \frac{1}{c^2}\vec{S} \ .$$

(a) (15 points) What are the frequency f, wavelength λ, and magnetic field $\vec{B}$ of the light field?

(b) (15 points) What is the time-average Poynting vector of the incident light?

(c) (30 points) What force does the light exert in the direction normal to the disk surface on the absorbing disk at a given angle θ? Similarly find the force exerted on the reflecting disk.

Hint: consider the total linear momentum change of the electromagnetic field due to the absorption and due to the reflection.

(d) (25 points) What is the total torque which is delivered by the light to the system of rod plus disks around the pivot point? What is the average torque over a full rotation of the rod?

(e) (15 points) Determine the approximate angular acceleration of the rod, averaged over a rotation. Recall that $R \gg r$ and the rod is massless.

Explicit Forms of Vector Operations

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian
$(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical
(ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \mathbf{e}_3 \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical
(r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1 \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right]$$
$$+ \mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3 \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)\right]$$

35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

	J	J	...
	M	M	...
m_1 m_2			
m_1 m_2	Coefficients		
. .			
. .			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2×1/2

m_1	m_2	1 (+1)
+1/2	+1/2	1

m_1	m_2	1 (0)	0 (0)
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	1 (−1)
−1/2	−1/2	1

1×1/2

m_1	m_2	3/2 (+3/2)
+1	+1/2	1

m_1	m_2	3/2 (+1/2)	1/2 (+1/2)
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	3/2 (−1/2)	1/2 (−1/2)
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	3/2 (−3/2)
−1	−1/2	1

2×1

m_1	m_2	3 (+3)
+2	+1	1

m_1	m_2	3 (+2)	2 (+2)
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	3 (+1)	2 (+1)	1 (+1)
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	3 (0)	2 (0)	1 (0)
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	3 (−1)	2 (−1)	1 (−1)
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	3 (−2)	2 (−2)
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	3 (−3)
−2	−1	1

1×1

m_1	m_2	2 (+2)
+1	+1	1

m_1	m_2	2 (+1)	1 (+1)
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	2 (0)	1 (0)	0 (0)
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	2 (−1)	1 (−1)
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	2 (−2)
−1	−1	1

3/2×1

m_1	m_2	5/2 (+5/2)
+3/2	+1	1

m_1	m_2	5/2 (+3/2)	3/2 (+3/2)
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	5/2 (+1/2)	3/2 (+1/2)	1/2 (+1/2)
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	5/2 (−1/2)	3/2 (−1/2)	1/2 (−1/2)
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	5/2 (−3/2)	3/2 (−3/2)
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	5/2 (−5/2)
−3/2	−1	1

2×1/2

m_1	m_2	5/2 (+5/2)
+2	+1/2	1

m_1	m_2	5/2 (+3/2)	3/2 (+3/2)
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	5/2 (+1/2)	3/2 (+1/2)
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	5/2 (−1/2)	3/2 (−1/2)
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	5/2 (−3/2)	3/2 (−3/2)
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	5/2 (−5/2)
−2	−1/2	1

3/2×1/2

m_1	m_2	2 (+2)
+3/2	+1/2	1

m_1	m_2	2 (+1)	1 (+1)
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	2 (0)	1 (0)
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	2 (−1)	1 (−1)
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	2 (−2)
−3/2	−1/2	1

QUALIFYING EXAMINATION, Part 2

1:00 pm – 5:00 pm, Friday September 2, 2016

Attempt all parts of all four problems.

Please begin your answer to each problem on a separate sheet, write your 3 digit code and the problem number on each sheet, and then number and staple together the sheets for each problem. Each problem is worth 100 points; partial credit will be given.

Calculators and cell phones may NOT be used.

Problem 1: Quantum Mechanics I

Consider a particle of mass m confined in one dimension to the region $-a \leq x \leq a$. This describes an infinite square well potential of width $2a$ centered at the origin.

(a) (20 points) Write down the normalized expressions for the even and odd parity solutions to the stationary Schrödinger equation, and find the corresponding energy levels $E_n\,(n = 1, 2, 3, \ldots)$ in terms of m, a and $\hbar$.

(b) (40 points) Let this system be perturbed by a delta function potential at the origin

$$H' = \beta E_1 a \delta(x),$$

where E_1 is the unperturbed ground-state energy, and β is a small dimensionless parameter. Find the corrections ΔE_n to the unperturbed energy levels through second order in β.

Useful formula: for n odd, $\sum 1/(n^2 - l^2) = -1/4n^2$, where the sum is over all positive odd integers $l \neq n$.

(c) (30 points) Find the exact energy eigenvalues in the presence of both the delta function potential and the infinite square well. To this end, use the piecewise solution for the wave function

$$\psi(x) = \begin{cases} A_+ \sin(k(x-a)) & \text{for} \quad 0 < x < a \\ A_- \sin(k(x+a)) & \text{for} \quad -a < x < 0\,. \end{cases}$$

Impose the continuity of $\psi(x)$ at $x = 0$, and determine the discontinuity of the wave function derivative $\psi'(x)$ at $x = 0$ by integrating the Schrödinger equation in a small interval around the origin. Consider the even and odd parity solutions separately, and show that the energy eigenvalues of one class of solutions are not affected by the presence of the delta function potential. For the other class, find the equation that determines implicitly the exact energy eigenvalues.

(d) (10 points) Check that the exact energy eigenvalues you found in (c) agree with your result in (b) to first order in β.

Problem 2: Quantum Mechanics II

In the classical (optical) Kerr effect, the index of refraction is proportional to the intensity. In quantum mechanics, we may use the following Hamiltonian to characterize the Kerr effect of the oscillator

$$\hat{H}_{Kerr} = \frac{\hbar\chi}{2}\left(a^{\dagger}\right)^2 a^2,$$

where $a^{\dagger}$ and a are the creation and annihilation operators of the bosonic excitations of the oscillator, and χ is the Kerr frequency shift per excitation.For simplicity neglect the oscillator Hamiltonian $\hat{H}_{osc} = \hbar\omega\left(a^{\dagger}a + 1/2\right)$.

The system starts at time $t = 0$ in a coherent state

$$|\psi(t=0)\rangle = |\alpha\rangle\,,$$

where $|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$ is a superposition of number eigenstates $a^{\dagger}a|n\rangle = n|n\rangle$, and α is a complex number.

(a) (20 points) Verify that the number state $|n\rangle$ is also the eigenstate of $\hat{H}_{Kerr}$ and compute its eigenvalue.

(b) (30 points) At time t, the system evolves to $|\psi(t)\rangle = e^{-i\hat{H}_{Kerr}t/\hbar}|\psi(0)\rangle$. Compute $|\psi(t)\rangle$ and its the overlap with the initial state $\langle\alpha|\psi(t)\rangle$. (You may keep the summation in the final expression.)

(c) (25 points) Find the smallest positive time $t_{revival}$ such that $|\psi(t_{revival})\rangle = |\alpha\rangle$. This is called the first revival of the coherent state.

(d) (25 points) At time $t = t_{revival}/2$, the system evolves to $|\psi(t_{revival}/2)\rangle$. Show that $|\psi(t_{revival}/2)\rangle$ can be written as a superposition of two coherent states, which is also called a “Schrödinger cat” state.

Hint: The following formulas might be useful

$$e^{i\frac{\pi}{2}n} = i^n$$

$$e^{-i\frac{\pi}{2}n^2} = \frac{e^{-i\pi/4} + (-1)^n e^{i\pi/4}}{\sqrt{2}}\,,$$

where n is an integer.

Problem 3: Statistical Mechanics I

Consider a spherical drop of liquid water containing N_l molecules surrounded by $N - N_l$ molecules of water vapor. The total number of molecules N is fixed, and we assume the temperature T is fixed at a given value throughout this problem. The drop and its surrounding vapor may be out of equilibrium.

(a) (15 points) Neglecting surface effects, write an expression for the Gibbs free energy of this system if the chemical potential of liquid water in the drop is μ_l and the chemical potential of water in the vapor is μ_v. Rewrite N_l in terms of the volume per molecule in the liquid v_l and the radius r of the drop. Note that v_l is constant independent of the size of the drop.

Hint: The contribution to the Gibbs free energy from a single phase with N_i molecules and chemical potential μ_i is $G_i = \mu_i N_i$.

(b) (15 points) The effect of the surface of the drop can be included by adding a contribution $G_{\text{surface}} = \sigma A$ to the Gibbs free energy in part (a), where σ is the surface tension ($\sigma > 0$) and A is the surface area of the drop. Write G_{total} with the surface contribution expressed in terms of r.

(c) (25 points) Make two qualitative sketches of G_{total} as a function of r: one sketch for $\mu_l - \mu_v > 0$ and a second sketch for $\mu_l - \mu_v < 0$. Starting from a drop with an arbitrary non-equilibrium radius, describe qualitatively how the size of the drop evolves in its approach to equilibrium in each of these two cases.

(d) (10 points) Under appropriate conditions, there is a critical radius r_c that separates drops which grow in size from those that shrink. Determine this critical radius.

(e) (35 points) Assume that the vapor behaves as an ideal gas and recall that the chemical potential of an ideal gas with pressure p is given by

$$\mu_v = \mu_v^0 + kT \ln(p/p^0) ,$$

where p^0 and μ_v^0 are, respectively, the pressure and chemical potential of a vapor in a reference state taken to be its equilibrium with a large flat surface of water. Write the chemical potential difference $\mu_l - \mu_v$ in terms of the vapor pressures p and p^0 and the temperature T. Derive the dependence of p/p^0 (known as the relative humidity) on the critical radius r_c.

Do drops grow more easily in an environment with a higher relative humidity or with a lower relative humidity? Explain your answer based on the expression you found for p/p^0.

Hint to part (e): Consider first the equilibrium condition between vapor and water in the reference state.

Problem 4: Statistical Mechanics II

Consider N non-interacting and non-relativistic electrons of mass m that are restricted to a 1D line of length L.

(a) (20 points) Determine the single-particle density of states $D(\epsilon)$ as a function of the single-particle energy ϵ.

(b) (15 points) Derive an expression for the Fermi energy ϵ_F of the system.

(c) (15 points) Determine the total energy per particle at temperature $T = 0$. Express your answer in terms of the Fermi energy.

(d) (30 points) Determine the temperature dependence of the chemical potential μ for this system to second order in T. The expansion coefficients should be expressed in terms of ϵ_F.

Hint: Use the following expansion

$$\int_0^\infty f(\epsilon)h(\epsilon)d\epsilon \approx \int_0^\mu h(\epsilon)d\epsilon + \frac{\pi^2}{6}(kT)^2 h'(\mu) + \dots ,$$

where $h(\epsilon)$ is a smooth function of ϵ, $f(\epsilon)$ is the Fermi-Dirac distribution at temperature T and chemical potential μ, and h' is the first derivative of h.

(e) (20 points) Does the chemical potential increase or decrease with temperature? Would your answer have changed if we had considered electrons in 3D? Provide a physical explanation for your answer.

Hint: Consider the behavior of the single-particle density of states as a function of ϵ.

Explicit Forms of Vector Operations

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian
$(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1\frac{\partial\psi}{\partial x_1} + \mathbf{e}_2\frac{\partial\psi}{\partial x_2} + \mathbf{e}_3\frac{\partial\psi}{\partial x_3}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical
(ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1\frac{\partial\psi}{\partial\rho} + \mathbf{e}_2\frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3\frac{\partial\psi}{\partial z}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \mathbf{e}_3\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical
(r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1\frac{\partial\psi}{\partial r} + \mathbf{e}_2\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right]$$
$$+\mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)\right]$$

35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			
.	.			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2×1/2

		1
		+1
+1/2	+1/2	1

		1	0
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		1
		−1
−1/2	−1/2	1

1×1/2

		3/2
		+3/2
+1	+1/2	1

		3/2	1/2
		+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

		3/2	1/2
		−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

		3/2
		−3/2
−1	−1/2	1

2×1

		3
		+3
+2	+1	1

		3	2
		+2	+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

		3	2	1
		+1	+1	+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

		3	2	1
		0	0	0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

		3	2	1
		−1	−1	−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

		3	2
		−2	−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

		3
		−3
−2	−1	1

1×1

		2
		+2
+1	+1	1

		2	1
		+1	+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

		2	1	0
		0	0	0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

		2	1
		−1	−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

		2
		−2
−1	−1	1

3/2×1

		5/2
		+5/2
+3/2	+1	1

		5/2	3/2
		+3/2	+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

		5/2	3/2	1/2
		+1/2	+1/2	+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

		5/2	3/2	1/2
		−1/2	−1/2	−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

		5/2	3/2
		−3/2	−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

		5/2
		−5/2
−3/2	−1	1

2×1/2

		5/2
		+5/2
+2	+1/2	1

		5/2	3/2
		+3/2	+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

		5/2	3/2
		+1/2	+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

		5/2	3/2
		−1/2	−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

		5/2	3/2
		−3/2	−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

		5/2
		−5/2
−2	−1/2	1

3/2×1/2

		2
		+2
+3/2	+1/2	1

		2	1
		+1	+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

		2	1
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		2	1
		−1	−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

		2
		−2
−3/2	−1/2	1