QUALIFYING EXAMINATION, Part 1

9:00 am – 11:30 am, Thursday August 30, 2018

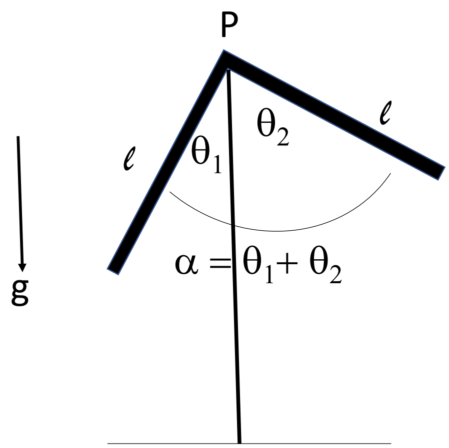
Attempt all parts of both problems.

Please begin your answer to each problem on a separate sheet, write your 3 digit code and the problem number on each sheet, and then number and staple together the sheets for each problem. Each problem is worth 100 points; partial credit will be given.

Calculators and cell phones may NOT be used.

Problem 1: Classical Mechanics I

Consider the V-shaped object in the figure below of mass M and two uniform rods of length l, with a fixed opening angle α. The object is supported at the pivot P and is subjected to a gravitational field $\vec{g}$.



(a) (15 points) Compute I, the moment of inertia of the object about an axis through P that is perpendicular to the plane of the rods.

In the following use the angle $\delta = \theta_1 - \frac{\alpha}{2}$ as generalized coordinate.

(b) (10 points) Find the kinetic energy T.

(c) (30 points) Find the gravitational potential energy V (with the heights measured from the pivot P).

(d) (5 points) Write down the Lagrangian L.

(e) (25 points) Find the equation of motion for δ.

(f) (15 points) Find the frequency of small oscillations in terms of M, l, and α.

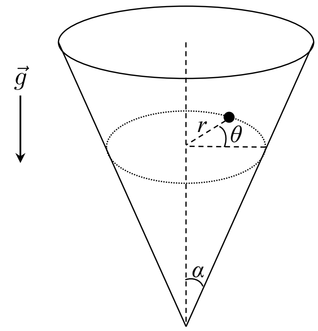
Useful formulas:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

Problem 2: Classical Mechanics II

A particle of mass m is constrained to move along the surface of a cone with half opening angle α, under the influence of gravity. The cone is oriented with its apex pointing down, and its symmetry axis along the direction of gravity $\vec{g}$.



(a) (20 points) Find the Lagrangian of the particle using (r,θ) as generalized coordinates, where r denotes the perpendicular distance from the cone axis to the particle, and θ is the azimuthal angle (see figure).

Hint: choose a cartesian coordinate system whose origin is the apex of the cone and whose z axis is the symmetry axis of the cone, and write the transformation equations expressing x, y, z in terms of r, θ. Alternatively, you can start with the general expression for T in cylindrical coordinates and restrict to the cone.

(b) (20 points) Calculate the generalized momenta, p_r and p_θ that are conjugate to the coordinates r and θ, respectively. Write down the Hamiltonian $H(r, \theta, p_r, p_\theta)$ in terms of the coordinates and conjugate momenta.

(c) (15 points) Using the Hamiltonian from part (b), find the four corresponding Hamilton's equations of motion for $\dot{r}, \dot{\theta}, \dot{p_r}$, and $\dot{p_\theta}$.

(d) (15 points) Determine which, if any, of the following quantities are conserved: p_r, p_θ, and H. Explain why the conservation follows from the form of the Hamiltonian, and provide a physical interpretation of any conserved quantities.

(e) (15 points) The particle can undergo motion along stable circular orbits around the cone. For an orbit with constant radius $r = r_0$, use the equations of motion from part (c) to find the corresponding value of p_θ (denote this value as L_0). Determine the corresponding angular frequency ω_0 of the circular orbit in terms of r_0, g, and α.

(f) (15 points) Assume the particle is perturbed slightly from the circular orbit in part (e), so that $r = r_0 + \delta r$ with $\delta r \ll r_0$. Find the frequency of small oscillations, ω, around r_0 for the perturbed orbit in terms of r_0, g, and α.

Hint: First find the equation of motion for $\ddot{r}$ for a given value of L_0, using either the results of parts (b) and (c), or directly from the Lagrangian in part (a). You can use the approximation $\frac{1}{(r_0+\delta r)^3} \approx \frac{1}{r_0^3}\left(1 - 3\frac{\delta r}{r_0}\right)$, and the value of L_0 found in (e) to simplify the equation of motion.

QUALIFYING EXAMINATION, Part 2

2:00 pm – 4:30 pm, Thursday August 30, 2018

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Problem 1: Electromagnetism I

A conducting spherical shell of radius a is split into two hemispheres, separated by an infinitesimally thin insulating layer. The upper hemisphere is maintained at a potential V, while the lower hemisphere is maintained at a potential $-V$.

(a) (15 points) What is the leading order radial dependence of the potential at $r \gg a$? Explain your reasoning. You do not need to determine the angular dependence or the proportionality constant for this part. Note that the total charge on the sphere (i.e., the sum of the charge of both hemispheres) is 0 by symmetry.

(b) (30 points) Find the potential $V(r, \theta)$ outside the shell ($r > a$). Make sure to explicitly specify which, if any, of the expansion coefficients A_l or B_l (see the general expression for V in the hint below) are zero. Determine the expansion only up to the lowest two non-zero terms; do not calculate the coefficients of the higher order terms.

(c) (15 points) Now consider the potential $V(r, \theta)$ inside the shell ($r < a$). Relate the coefficients A_l and B_l inside the shell to the coefficients A_l and B_l outside the shell.

(d) (20 points) Determine the potential $V(r, \theta)$ inside the shell ($r < a$). Calculate only the lowest two non-zero terms in the expansion.

(e) (20 points) Calculate the net surface charge density $\sigma(\theta)$ on the shell up to the lowest two non-zero terms.

Hints: The general solutions of Laplace's equation with azimuthal symmetry in spherical coordinates (where the z axis is taken to be the symmetry axis) is

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta) \ ,$$

where P_l are Legendre polynomials and A_l, B_l are constants.

You may find the following properties of the Legendre polynomials useful

$$\int_{-1}^{1} dx P_l(x) P_m(x) = \frac{2}{2l+1} \delta_{lm}$$

$$P_l(-x) = (-1)^l P_l(x)$$

$$\int_0^1 dx\, P_0(x) = 1 \ , \quad \int_0^1 dx\, P_1(x) = 1/2 \ , \quad \int_0^1 dx\, P_2(x) = 0 \ , \quad \int_0^1 dx\, P_3(x) = -1/8 \ .$$

Problem 2: Electromagnetism II

A capacitor consists of a pair of parallel circular perfectly conducting plates both of radius a. The lower plate lies on the plane $z = 0$, while the second plate is at $z = d \ll a$ so that edge effects can be ignored. Both plates are centered on the z-axis.

The lower plate has a uniform time-dependent surface charge $\sigma(t) = \sigma_0 \cos \omega t$. In the following assume the quasi-static approximation, $a\omega/c \ll 1$, in which retardation effects are neglected.

(a) (15 points) Calculate the time-dependent electric field $\mathbf{E}$ between the plates.

(b) (25 points) Calculate the magnetic field $\mathbf{B}$ induced by the displacement current of the electric field in part (a). The displacement current is given by $\mathbf{J}_d = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ (SI units) or $\mathbf{J}_d = \frac{1}{4\pi} \frac{\partial \mathbf{E}}{\partial t}$ (Gaussian units).

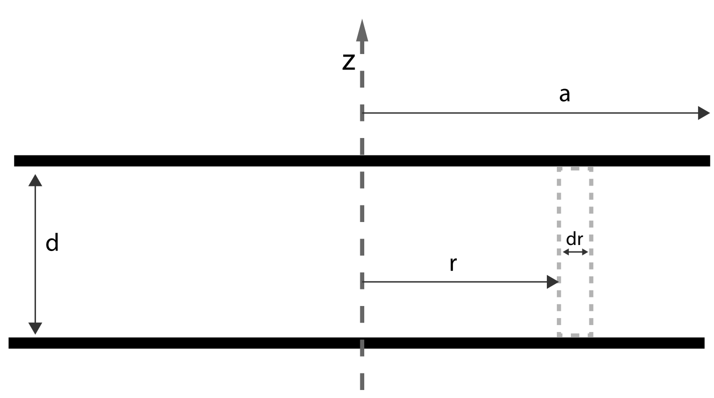
(c) (15 points) Compute the Poynting vector $\mathbf{S}$ and the energy flux through a cylindrical surface $x^2 + y^2 = r^2$ $(r < a)$ that lies inside the capacitor between $z = 0$ and $z = d$.

The Poynting vector is given by $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$ (SI units), or $\mathbf{S} = \frac{c}{4\pi}(\mathbf{E} \times \mathbf{B})$ (Gaussian units), where c is the speed of light.

(d) (25 points) Calculate the rate of change of electromagnetic energy in the volume enclosed by the cylindrical surface in (c). Consider only the term that is first order in ω. Compare your result with the energy flux you found in (c).

(e) (20 points) The induced magnetic field in (b) generates an additional induced electric field $\mathbf{E}_{\text{ind}}(r) = E_{\text{ind}}(r)\,\hat{\mathbf{z}}$ which is second order in ω. Calculate $E_{\text{ind}}(r) - E_{\text{ind}}(r = 0)$.

Hint: consider the rectangular Faraday's loop with sides d and dr shown in the figure. The figure shows a cross section of the system using a plane perpendicular to the plates that goes through their center.



QUALIFYING EXAMINATION, Part 3

9:00 am – 11:30 am, Friday August 31, 2018

Attempt all parts of both problems.

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Problem 1: Quantum Mechanics I

Consider a non-relativistic particle of mass m moving in one dimension along the x axis in a potential $V(x)$. The particle is described by normalized wave function $\psi(x,t)$ that satisfies the time-dependent Schrödinger equation.

(a) (30 points) The Schrödinger equation conserves global probability so that $\int_{-\infty}^{\infty} dx\, |\psi(x,t)|^2$ remains constant for all times t. However, it also preserves probability locally. The probability current is defined by

$$J(x,t) = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right),$$

and we denote the probability to find the particle in the interval $a < x < b$ at time t by $P_{ab}(t)$. Derive an expression for dP_{ab}/dt in terms of the probability current $J(x,t)$ at the two end points $x = a$ and $x = b$.

(b) (25 points) Now consider the following wave function

$$\psi(x,t) = A\exp(-c|x| - iEt/\hbar)\ ,$$

where $A > 0$, $c > 0$ and E are constants. Determine A. Then find the expectation values $\langle\hat{x}\rangle(t)$ and $\langle\hat{x}^2\rangle(t)$.

Hint:

$$\int_0^\infty du\, u^n \exp(-u) = n! \quad \text{for } n > -1\,.$$

(c) (25 points) For an arbitrary Hermitian observable $\hat{O}$ that does not depend explicitly on time, determine the expectation value $\langle\hat{O}\rangle(t)$ of $\hat{O}$ at time t in terms of its expectation value at time $t = 0$ for the wave function of part (b). Explain briefly why the result looks the way it does.

(d) (20 points) Using the Schrödinger equation, find the potential $V(x)$ appropriate for the wave function $\psi(x,t)$ of part (b). Assume that $V(x) \to 0$ as $x \to \pm\infty$.

Hint: consider first the Schrödinger equation for $x > 0$ and $x < 0$ and determine the potential for $x \neq 0$. Next, integrate the Schrödinger equation in a narrow interval $-\epsilon < x < \epsilon$ around $x = 0$ and use the explicit form of the wave function in (b).

Problem 2: Quantum Mechanics II

The Stark Effect in atomic hydrogen is the shift in energies when the atom is placed in an external, uniform electric field. With the electric field pointed in the z direction, the perturbing potential is

$$V = -e\epsilon z \ ,$$

where ϵ is the strength of the electric field, and where the proton is situated at the origin.

(a) (15 points) Neglecting the spin of the electron, the unperturbed bound-state energy of the hydrogen atom depends on only the principal quantum number n. For the $n = 2$ level, how many degenerate states exist? Label them by the orbital angular momentum $l = 0$ ($2s$ states) and $l = 1$ ($2p$ states), with the associated values of the magnetic quantum number m_l in each case.

(b) (25 points) Apply degenerate perturbation theory to the states in (a) to determine the energy shifts to first (linear) order in ϵ. To identify the non-vanishing matrix elements of V, take into account the parity properties of the unperturbed states and of V. Also take into account the fact that since the potential is the zero component of a spherical tensor of rank one, it can connect only states with the same value of m_l.

Useful matrix element: $\langle 2s|V|2p, m_l = 0\rangle = 3e\epsilon a_0$, where a_0 is the Bohr radius.

(c) (20 points) What are the zeroth-order states that correspond to the shifted energies you found in (b)? Express them as linear combinations of $2s$ and $2p$ states labeled with appropriate values of m_l.

Next take into account the spin of the electron. The degenerate, unperturbed states can be denoted by $2p_{j=3/2}$, $2p_{j=1/2}$, and $2s_{j=1/2}$, along with their appropriate m_j values. The spin-orbit splitting lifts the $2p_{3/2}$ states above the other states ($2p_{1/2}$ and $2s_{j=1/2}$). Assume that $e\epsilon a_0$ is much smaller than the spin-orbit splitting. Thus the $2p_{3/2}$ states can be ignored here. The remaining degeneracy between the $2p_{1/2}$ and $2s_{1/2}$ states is also broken, by the much smaller Lamb shift, which splits these states by a small energy δ. Examine the Stark shift among the $2s_{1/2}$ states and the $2p_{1/2}$ states.

(d) (20 points) First consider the limit $e\epsilon a_0 \gg \delta$. Which are the non-vanishing matrix elements of V among the $2s_{1/2}$ and $2p_{1/2}$ states? Here, the tensor property of V, as described in part (b), insures that it can connect only states with the same value of m_j. How does the Stark shift depend on ϵ in this limit?

(e) (20 points) In the opposite limit $e\epsilon a_0 \ll \delta$, how does the Stark shift of each of the two degenerate levels, $2s_{1/2}$ and $2p_{1/2}$, depend on ϵ? Why? Can this behavior be determined by non-degenerate perturbation theory?

QUALIFYING EXAMINATION, Part 4

2:00 pm – 4:30 pm, Friday August 31, 2018

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Problem 1: Statistical Mechanics I

Consider a *classical* non-relativistic gas of N non-interacting identical particles with mass m moving in one dimension, constrained by a wall at $x = a$ to move in the $x \geq a$ region and subject to the potential

$$V(x) = bx \quad \text{for} \;\; x \geq a \; ,$$

where $b > 0$. The gas is in equilibrium at uniform temperature T.

(a) (35 points) Compute the partition function Z and the free energy F of the gas.

Hint: the classical partition function z for a *single* particle is given by

$$z = \int \frac{dx\, dp}{2\pi\hbar}\, e^{-\beta E(p,x)} \; ,$$

where $\beta = \frac{1}{k_B T}$ and $E(x, p)$ is the energy of the particle.

Useful formula:

$$\int_{-\infty}^{\infty} e^{-\lambda x^2} \mathrm{d}x = \sqrt{\frac{\pi}{\lambda}}$$

(b) (25 points) Use the free energy you found in (a) to determine the force f that the gas exerts on the wall.

Hint: consider the work done by the gas at constant temperature T when the wall is moved by a small distance da.

(c) (25 points) Find the density of particles $n(x)$ as a function of x for $x \geq a$.

(d) (15 points) Express the force f you found in (b) in terms of $n(x = a)$ and T. Would the relation between f and $n(a)$ change if the confining potential has a different form? Justify or provide a derivation of your answer.

Problem 2: Statistical Mechanics II

Consider a one-dimensional gas of N non-interacting identical non-relativistic fermions of mass m confined in a harmonic trap potential $V(x) = \frac{1}{2}m\omega^2 x^2$, where ω is the trap frequency. The gas is in equilibrium at temperature T. For simplicity, assume a single-spin species, i.e., ignore the spin degeneracy of each level.

(a) (15 points) Find the Fermi energy ϵ_{F} of the gas at $T = 0$ as a function of N.

(b) (25 points) (i) Calculate the exact total energy per particle E/N at $T = 0$.

Useful formula:

$$\sum_{n=0}^{N-1} n = \frac{N(N-1)}{2} \ .$$

(ii) Determine the *average* single-particle density of states in the harmonic trap, and use it to calculate the average total energy per particle. Compare with your result in (i).

(c) (20 points) Write down an expression for the grand potential $\Omega(\mu, T) = -k_{\mathrm{B}}T \ln Z_{\mathrm{GC}}$, where k_{B} is the Boltzmann constant, Z_{GC} is the grand-canonical partition function of the gas and μ is the chemical potential. Do not attempt to evaluate the infinite sum.

(d) (20 points) Use your result in (c) to derive the equation of state (EoS) of the gas, i.e., find the function $\bar{N}(\mu, T)$, where $\bar{N}$ is the average total number of particles. Do not attempt to evaluate the infinite sum.

(e) (20 points) (i) Calculate the EoS you found in (d) in the limit of high T ($kT \gg \hbar\omega$) and $z \ll 1$, where $z = e^{\beta\mu}$ is the fugacity of the gas. Evaluate explicitly the infinite sum.

(ii) What does the limit in (i) describe?