QUALIFYING EXAMINATION, Part 1

10:00 am – 12:30 pm, Thursday August 29, 2019

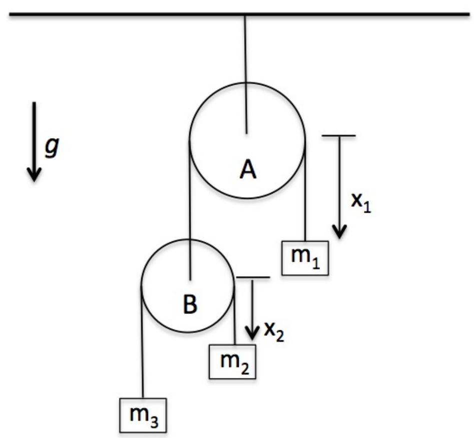
Attempt all parts of both problems.

Please begin your answer to each problem on a separate sheet, write your 3 digit code and the problem number on each sheet, and then number and staple together the sheets for each problem. Each problem is worth 100 points; partial credit will be given.

Calculators and cell phones may NOT be used.

Problem 1: Classical Mechanics I

A compound Atwood machine is composed of three masses m_1, m_2, and m_3 attached to two massless ropes through two massless pulleys A and B of radii r_A and r_B, respectively (see figure). Pulley A is attached to a stationary ceiling. The lengths l_a and l_b of the ropes around pulleys A and B are fixed, and the ropes do not slip as the pulleys rotate. The system is in a constant gravitational field with acceleration g.



Take as two generalized coordinates x_1 and x_2 the distance from the center of pulley A to mass m_1 and the distance from the center of pulley B to mass m_2, respectively (see figure).

(a) (25 points) Write down the vertical positions x_{m_1}, x_{m_2}, and x_{m_3} of the three masses (as measured from the center of pulley A) in terms of x_1 and x_2, and find the total kinetic energy T.

(b) (20 points) Find the gravitational potential energy V. Set the potential energy to zero at the center of pulley A.

(c) (5 points) Write down the Lagrangian L of the system.

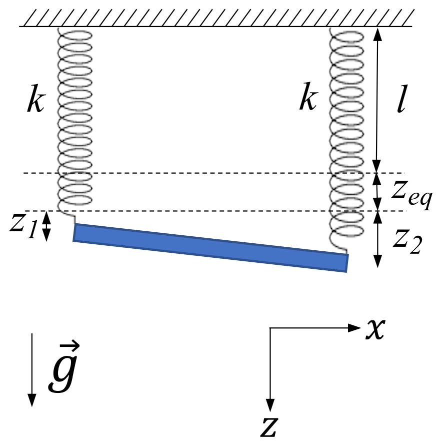
(d) (30 points) Derive the equations of motion for x_1 and x_2. You do not need to solve them explicitly.

(e) (20 points) Next assume that pulley A is a uniform disk with mass M (and radius r_A). Write down the Lagrangian L for this system.

Hint: the rope does not slip on the pulley.

Classical Mechanics II

A bar of length d and mass M hangs under the influence of gravity from two springs with equal unstretched lengths, l, and equal spring constants, k, as shown in the figure.



In the following parts, consider only motion of the bar in the x-z plane and ignore any "swinging" motions—i.e., consider only modes of small oscillations where the center of the bar does not move in the x or y directions. In these modes, the bar center and ends are free to move primarily in the z direction, including rotations in the x-z plane.

(a) (10 points) Find the equilibrium position of the bar, z_{eq}, relative to the position of the ends of the unstretched springs.

For small oscillations around this equilibrium position, two independent modes are possible:

1. A mode in which the center-of-mass oscillates in the z direction with the bar horizontal
2. A mode in which the center-of-mass is fixed and the bar rotates in the x-z plane

(b) (30 points) Write down the equations of motion for the bar in each independent mode by considering the corresponding forces and torques on the bar. What are the frequencies of the oscillations in each of the modes?

Next determine the modes in a more systematic way, without the need to guess the independent degrees of freedom:

(c) (20 points) Calculate the kinetic energy, T, and potential energy, V, in terms of the displacements z_1 and z_2 from the equilibrium position for small oscillations.

(d) (20 points) Using the results of part (c), find the eigenfrequencies of the two independent modes.

(e) (20 points) Find the normal modes corresponding to these frequencies and check that they agree with the intuition about the independent degrees of freedom used in part (b).

QUALIFYING EXAMINATION, Part 2

2:00 pm – 4:30 pm, Thursday August 29, 2019

Attempt all parts of both problems.

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Problem 1: Electromagnetism I

Consider a sphere of radius R and negative charge $-q$ distributed uniformly within its volume. The sphere is centered at the origin of our coordinate system.

(a) (30 points) Find the electric field and electric potential outside the sphere ($r > R$) and inside the sphere ($r < R$).

(b) (20 points) Sketch the electric field and potential of part (a) as a function of r.

Assuming that the spherical charge density above remains fixed in space, we add a positive point charge $+q$ that is free to move under the influence of the electric field produced by the sphere (this can be used to model a classical polarizable atom).

(c) (5 points) Identify the position of stable equilibrium for the point charge.

(d) (15 points) Next, we perturb the system by applying a uniform electric field, $\mathbf{E} = E_0\hat{\mathbf{x}}$. Assuming the point charge moves a distance that is smaller than the radius of the sphere R, find its new stable equilibrium position.

(e) (15 points) Find the dipole moment $\mathbf{p}$ of the perturbed system in (d) which includes both the point charge and the sphere.

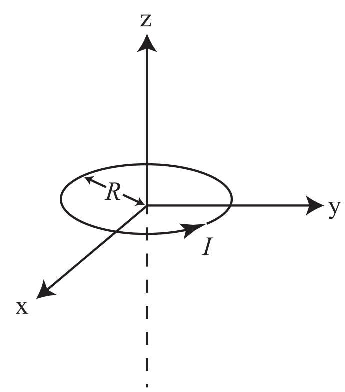
(f) (15 points) Find the effective polarizability α of the perturbed system.

Hint: The dipole moment $\mathbf{p}$ is related to the applied electric field by $\mathbf{p} = \alpha\mathbf{E}$.

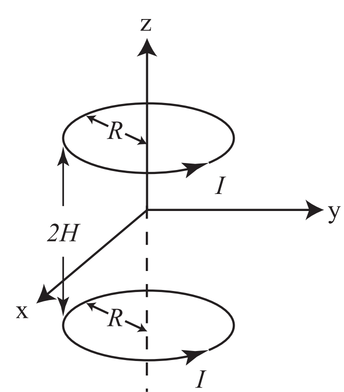
Problem 2: Electromagnetism II

(a) (20 points) Calculate the magnetic field along the symmetry axis z of a loop of radius R carrying a constant current I_0 (see figure).

Hint: Biot-Savart law states $\mathrm{d}\mathbf{B} = \mu_0 I \,\mathrm{d}\mathbf{l} \times \hat{\mathbf{r}}/(4\pi r^2)$ in SI units, and $\mathrm{d}\mathbf{B} = I \,\mathrm{d}\mathbf{l} \times \hat{\mathbf{r}}/(cr^2)$ in Gaussian units.



(b) (15 points) Two identical loops of radius R are separated by a distance $2H$ (along the symmetry axis z) and each carries a constant current I_0 flowing in the same direction (see figure). Find the magnetic field on axis between the two loops as a function of z, where $z = 0$ is the midpoint between the two loops.



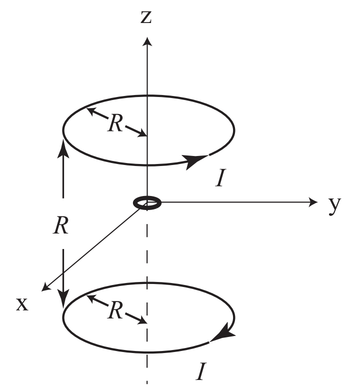
(c) (10 points) At a separation of $H = R/2$, the magnetic field is close to being uniform in the vicinity of $z = 0$ (Helmholtz). Sketch the magnetic field lines along the z axis and off-axis.

(d) (15 points) We reverse the direction of the current in the bottom loop of the system of part (c) (Anti-Helmholtz). Sketch the magnetic field lines along the axis and off-axis.

(e) (20 points) The magnetic field along the z axis near the center of the system in part (d) can be shown to be $B_z = \eta I_0 z$, where η is a constant (do not derive this result). Find $\partial B_x/\partial x$ and $\partial B_y/\partial y$ at the center of the system (i.e., at $x = y = z = 0$).

Hint: use one of Maxwell's equation and the cylindrical symmetry of the system.

(f) (20 points) Next we place a small loop of wire of resistance Z_0 and radius $r \ll R$ along the symmetry axis near the center of the system (see figure), and vary the current in the top (bottom) loop as $I = I_0 \cos(\omega_0 t)$ ($I = -I_0 \cos(\omega_0 t)$). Find the magnitude and direction of the current induced in the small loop as a function of z.



QUALIFYING EXAMINATION, Part 3

9:00 am – 11:30 am, Friday August 30, 2019

Attempt all parts of both problems.

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Problem 1: Quantum Mechanics I

Part 1:

Consider the following two operators in the three-dimensional Hilbert space of a particle

$$\hat{L}_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad \hat{L}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{1}$$

(a) (20 points) Find the eigenvalues of $\hat{L}_x$ and its normalized eigenvector with the highest eigenvalue.

(b) (15 points) What is the eigenstate of $\hat{L}_z$ that corresponds to its highest eigenvalue? Calculate $\langle \hat{L}_x \rangle$ and $\langle \hat{L}_x^2 \rangle$ in this state.

(c) (15 points) If $\hat{L}_z$ is measured in the eigenstate of $\hat{L}_x$ with the highest eigenvalue, what are the probabilities of obtaining the various possible outcomes?

Part 2:

Consider the harmonic oscillator Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 \ ,$$

where $\hat{x}$ and $\hat{p}$ are, respectively, the position and momentum operators.

(d) (20 points) Using the formulas given in the hints below, find the uncertainty Δx in the eigenstate $|n\rangle$ with energy $E_n = \left(n + \frac{1}{2}\right) \hbar\omega$.

(e) (15 points) Find an unnormalized ground-state wavefunction $\psi_0(x)$ in coordinate space using that the corresponding ket $|0\rangle$ obeys $a|0\rangle = 0$, where a is the annihilation operator.

(f) (15 points) Find an unnormalized ground-state wavefunction $\psi_0(p)$ in momentum space.

Hints:

$$a = \left(\frac{m\omega}{2\hbar}\right)^{1/2} \hat{x} + i \left(\frac{1}{2m\omega\hbar}\right)^{1/2} \hat{p}$$

$$a|n\rangle = \sqrt{n}|n-1\rangle \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

Problem 2: Quantum Mechanics II

Approximate the ammonia molecule NH_3 by a simple two-state system. The three H nuclei are in a plane and the N nucleus is at some fixed distance either above or below the plane of the H's. Each of the two states is approximately an eigenstate (a normalized up state $\binom{1}{0}$ and a normalized down state $\binom{0}{1}$) with the same energy E_0. However, there is a small amplitude A for transition from the up state to the down state. Thus the Hamiltonian of the two-state system can be written as $H = H_0 + H'$, where

$$H_0 = \begin{pmatrix} E_0 & 0 \\ 0 & E_0 \end{pmatrix} ,$$

and

$$H' = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} .$$

Take A to be real with $|A| \ll |E_0|$.

(a) (25 points) Find the exact eigenvalues of H. What are the corresponding normalized eigenstates in terms of the up state $\binom{1}{0}$ and the down state $\binom{0}{1}$?

(b) (25 points) Next suppose that the molecule is placed in an electric field that distinguishes the up and down states. The new Hamiltonian is $H = H_0 + H' + H''$, where

$$H'' = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}$$

with $\epsilon_1 \neq \epsilon_2$. What are the exact energy eigenvalues of the new Hamiltonian H?

(c) (20 points) When $|\epsilon_1|, |\epsilon_2| \ll |A|$, what answer does time-independent perturbation theory give for the energies of H to lowest non-vanishing order in ϵ_i? Compare your result to the exact answer.

(d) (30 points) When $|A| \ll |\epsilon_1|, |\epsilon_2|, |\epsilon_2 - \epsilon_1|$, what answer does time-independent perturbation theory give for the energies of H to lowest non-vanishing order in A? Compare your result to the exact answer.

QUALIFYING EXAMINATION, Part 4

2:00 pm – 4:30 pm, Friday August 30, 2019

Attempt all parts of both problems.

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Calculators and cell phones may NOT be used.

Problem 1: Statistical Mechanics I

Consider a non-interacting classical two-dimensional (2D) gas of N non-relativistic identical particles confined in a harmonic trap with hard walls at $r = R$

$$V_{\text{trap}}(r) = \begin{cases} \frac{1}{2}m\omega^2 r^2 & \text{for } r < R \\ \infty & \text{for } r \geq R \end{cases},$$

where m is the particles mass, ω is the trap angular frequency and r is the radial coordinate.

(a) (20 points) Show that the density distribution $n(r)$ of this gas in thermal equilibrium at temperature T is $n(r) = Ce^{-\beta V_{\text{trap}}(r)}$, with $\beta^{-1} = k_{\text{B}}T$. Compute C such that $n(r)$ is normalized as $\int n(r)\mathrm{d}^2\mathbf{r} = N$. Sketch $n(r)/n(0)$ as a function of r/R.

Next, this gas is set to rotate at a fixed angular velocity Ω around the z axis, defined as perpendicular to the gas plane and intersecting it at the center of the trap $\mathbf{r} = 0$. The thermodynamics of a non-interacting gas in equilibrium in the rotating frame can be calculated by replacing the single-particle Hamiltonian in the lab frame $h_{\text{lab}} = \mathbf{p}^2/(2m) + V_{\text{trap}}(r)$ by $h_{\text{rot}} = h_{\text{lab}} - \mathbf{\Omega} \cdot \mathbf{L}$, where $\mathbf{\Omega} = \Omega\,\hat{\mathbf{z}}$ ($\hat{\mathbf{z}}$ is the unit vector along the z axis) and $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the angular momentum of the particle (note that $\mathbf{L}$ is collinear with $\hat{\mathbf{z}}$ here).

(b) (25 points) Show that h_{rot} can be written in the form $h_{\text{rot}} = \frac{1}{2m}(\mathbf{p} - m\mathbf{\Omega} \times \mathbf{r})^2 + V_{\text{eff}}(r)$, where $V_{\text{eff}}(r) = \frac{1}{2}m(\omega^2 - \Omega^2)r^2$ for $r < R$ (and ∞ otherwise), and interpret this result.

(c) (20 points) Suppose the rotating gas is in equilibrium at a temperature T. Show that its density distribution is $n_\Omega(r) = De^{-\beta V_{\text{eff}}(r)}$, and calculate D to normalize this distribution as in (a). For simplicity, express your result in terms of the parameter $\alpha = \frac{m(\omega^2 - \Omega^2)}{2k_{\text{B}}T}R^2$. Sketch $n_\Omega(r)/n_\Omega(0)$ as a function of r/R for slow ($\Omega \ll \omega$), fast ($\Omega = \omega$), and ultrafast rotation ($\Omega \gg \omega$).

(d) (15 points) Show that the canonical partition function of the classical gas has the form $\mathcal{Z} = \mathcal{Z}_0(1 - e^{-\alpha})^N$ where $\mathcal{Z}_0$ is a proportionality factor independent of R (you do not need to calculate $\mathcal{Z}_0$).

(e) (20 points) Using the partition function in (d), calculate the force $F = k_{\text{B}}T\frac{\partial \log \mathcal{Z}}{\partial R}$ exerted by the gas on the disk walls, and thus the pressure $P = \frac{F}{2\pi R}$. Express your results for $\frac{P\pi R^2}{Nk_{\text{B}}T}$ as a function of α.

Find P for fast rotation ($\Omega = \omega$) and its asymptotic form in the limit of ultrafast rotation ($\Omega \gg \omega$), and interpret your results.

Useful reminders:

The canonical partition function of a classical gas of N identical particles is
$\mathcal{Z} = \frac{1}{h^{2N}N!}\left(\int \mathrm{d}^2\mathbf{r}\,\mathrm{d}^2\mathbf{p}\; e^{-\beta h(\mathbf{r},\mathbf{p})}\right)^N$, where $h(\mathbf{r},\mathbf{p})$ is the single-particle Hamiltonian.

$\int_0^R re^{-\lambda r^2}\mathrm{d}r = \frac{1-e^{-\lambda R^2}}{2\lambda}$ (recall the Jacobian in polar coordinates is $\mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\theta$)

$\int_{-\infty}^{\infty} e^{-\lambda u^2}\mathrm{d}u = \sqrt{\frac{\pi}{\lambda}}$

$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$

Problem 2: Statistical Mechanics II

Consider non-interacting spin-1/2 fermions in two dimensions (2D) with a linear dispersion relation

$$\epsilon_\pm(\vec{k}) = \pm\hbar v|\vec{k}| \; .$$

Positive energy states (with energy ϵ_+) define the *conduction* band and negative energy states (with energy ϵ_-) define the *valence* band. Assume that the allowed wavevectors $\vec{k} = \{k_x, k_y\}$ correspond to periodic boundary conditions over a square region of area A.

At temperature $T = 0$ the valence band is completely filled and the conduction band is completely empty. At finite T, excitations above this ground state correspond to adding *particles* (occupied states) in the conduction band or *holes* (unoccupied states) in the valence band.

(a) (25 points) Find the single-particle density of states $D(\epsilon)$ as a function of the energy ϵ in terms of $\hbar, v, A$. Sketch $D(\epsilon)$ over both the negative and positive energy region.

In the next two parts we will argue that the chemical potential $\mu(T) = 0$ at any temperature T (which can be assumed so for the rest of the problem).

(b) (20 points) Using the Fermi-Dirac distribution, show that if $\mu(T) = 0$ then the probability of finding a particle at energy ϵ is equal to the probability of finding a hole at energy $-\epsilon$.

(c) (15 points) Particle number conservation requires that the number of particles in the conduction band must equal the number of holes in the valence band, $N_p = N_h$. Show that if $\mu(T) = 0$ then this condition holds at all T. (Do not worry if the integrals are formally divergent – they will be cut off in any physical system.)

(d) (25 points) Find the total internal energy of the excitations above the $T = 0$ state, $U(T) - U(0)$, expressed in terms of $A, v, \hbar, k_B$. Note that since we are subtracting $U(0)$, in the valence band you only need to count the energy associated with holes.

Useful integral:

$$\frac{1}{n!}\int_0^\infty \frac{x^n}{e^x + 1} = (1 - 2^{-n})\zeta(n+1)\,,$$

where $\zeta(n+1) = \sum_{k=1}^{\infty} k^{-n-1}$ is the Riemann zeta function.

(e) (15 points) Use your answer in (d) to find the heat capacity at constant area $C_A(T) \propto T^\alpha$. What is the exponent α?